2.6 The microscope equation

As we saw in Section 2.2, the graph of a function $y = f(x)$ near a point $x = a$ is similar to the graph of the tangent line to $f(x)$ at that point, if $f'(a)$ exists. In this section, we use this observation to perform "linear approximation."

We begin by recalling (cf. equation (2.2.4)) the following:

If $f(x)$ is locally linear at $x = a$, then

$$f(a + \Delta x) \approx f(a) + f'(a)\Delta x \text{ for } \Delta x \text{ small enough.}$$

The microscope equation

It's called "the microscope equation" because it tells us what a function $y = f(x)$, locally linear at a point $x = a$, looks like "under a microscope" focussed on that point: it looks like its tangent line there!

The geometry of the situation is captured in the following picture.

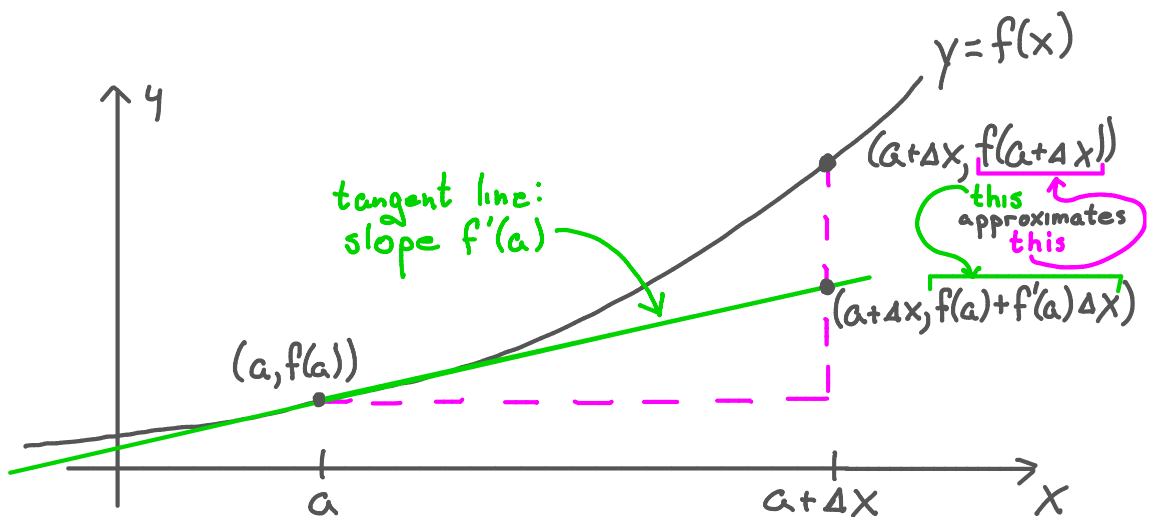

Figure 2.7. The microscope equation

The utility of the microscope equation, for the purposes of the present section, is as follows. Sometimes, one has explicit information about a function and its derivative *at* a particular point $x = a$, but this function is less concretely understood at nearby points. The microscope equation transforms the known information at the given point into approximate information about what happens in the vicinity.

Example 2.6.1. Use the microscope equation to approximate $\sqrt{65}$.

Solution. The idea is this: $a = 64$ is an "easy" input for the square root function $f(x) = \sqrt{x}$: both the function value $f(a) = f(64) = \sqrt{64} = 8$ and the derivative value $f'(a) = f'(64) = \frac{1}{2} \times 64^{-1/2} =$

1/16 are known, and are "simple." We will use this information to get a good approximation to the value of this function at the nearby, but less "simple," input $x = 65$.

To do this, we note that $65 = 64 + 1$, so we can write $65 = a + \Delta x$ where, again, $a = 64$, and $\Delta x = 1$. Then

$$\sqrt{65} = f(65) = f(a + \Delta x) \approx f(a) + f'(a)\Delta x = \sqrt{64} + \frac{1}{16} \times 1 = 8.0625.$$

To summarize: by the microscope equation, $\sqrt{65} \approx 8.0625$.

Of course, we could have approximated $\sqrt{65}$ more naively: we might have argued that 65 is close to 64, so the square root of 65 should be close to the square root of 64, so the square root of 65 should be about 8. But note that this does not give nearly as good an approximation. In fact, a calculator gives the "true" answer $\sqrt{65} = 8.0622....$ So our linear approximation is off by less than three ten-thousandths, whereas the estimate $\sqrt{65} \approx 8$ is more than six hundredths away from the actual value.

Regarding the above example, we make two observations. First: we needed to know $f'(64)$. In general, the microscope equation – that is, linear approximation – works only when we have information about a function *and* its derivative at a particular point.

Second: wouldn't it have been better to just use a calculator in the first place? It gives a better result and is much easier! Well yes, this is all true. BUT: your calculator itself uses, in essence, the microscope equation! More specifically: machine calculations of things like $\sqrt{65}$, $\cos(\pi/7)$, $3^{\tan(0.2)}$, and other quantities that aren't as "simple" as $\sqrt{64}$ often invoke algorithms that amount to linear approximation.

Of course technology does, as noted above, generally, provide more accurate results than one might obtain through the method of Example 2.6.1. But this is not necessarily because technology uses a *different* method, it's because technology applies this method *repeatedly*.

To explain this, let's suppose we *don't* have a calculator, and we *don't* know anything about $\sqrt{65}$, other than the approximation 8.0625 obtained above. Let's write e for the *error* in this approximation. That is, $e = 8.0625 - \sqrt{65}$. Again, we don't know $\sqrt{65}$ exactly, so we don't know e exactly. But we can *approximate* e using (essentially) the microscope equation! Adding this approximation of e to our original estimate of $\sqrt{65}$ gives us a *better* estimate of this square root. And we can repeat this process over and over, obtaining a better approximation each time.

This process, and variants of it, are at the heart of "machine" computation of quantities involving more than just the usual addition, multiplication, and so on.

Further, repeated application of the microscope equation is how such quantities were approximated back in the day, before such machines existed.

Often, when implementing the microscope equation, we want to specify our function $f(x)$ and our "good" point $x = a$, but not necessarily our Δx. Doing this will provide us with a recipe for approximating $f(x)$ at *any* point "near" $x = a$.

Example 2.6.2. Let $g(x) = 1/(1 + x)^3$.

(a) Write down the microscope equation for $g(x)$ at $x = 0$.

(b) Use this result to estimate $1/0.99^3$ and $1/1.03^3$.

Solution. (a) Here our "good" point is $a = 0$. We have

$$g(a) = g(0) = \frac{1}{(1+0)^3} = 1; \quad g'(a) = g'(0) = \frac{-3}{(1+0)^4} = -3.$$

We are not yet choosing a particular Δx, so for the moment, we let Δx be an arbitrary number (though we think of Δx as being "small"). Then the microscope equation tells us:

$$\begin{aligned} g(a + \Delta x) &\approx g(a) + g'(a)\Delta x \\ g(0 + \Delta x) &\approx g(0) + g'(0)\Delta x \\ g(\Delta x) &\approx 1 - 3\Delta x \\ \frac{1}{(1+\Delta x)^3} &\approx 1 - 3\Delta x. \end{aligned} \tag{2.6.1}$$

(b) To estimate these numbers, we apply the above result with appropriate choices for Δx. For the first quantity, we choose $\Delta = -0.01$, since this turns the left-hand side of equation (2.6.1) into $1/0.99^3$. So that equation tells us

$$\frac{1}{0.99^3} = \frac{1}{(1+\Delta x)^3} \approx 1 - 3\Delta x = 1 - 3(-0.01) = 1.03.$$

Similarly, with $\Delta = 0.03$, equation (2.6.1) gives

$$\frac{1}{1.03^3} = \frac{1}{(1+\Delta x)^3} \approx 1 - 3\Delta x = 1 - 3(0.03) = 0.91.$$

A calculator gives $1/0.99^3 = 1.03061...$ and $1/1.03^3 = 0.91514....$ So the first estimate is closer to the true value than is the second. This is because the first estimate entails a Δx of smaller magnitude ($|\Delta x| = 0.01$ as opposed to $|\Delta x| = 0.03$). Generally speaking, the smaller Δx, the better an approximation the microscope equation affords.

Here is a microscope equation that also involves the chain rule.

Example 2.6.3. (a) Write down the microscope equation for $f(x) = \sqrt{4 + \sin(x)}$ at $x = 0$.

(b) Approximate $\sqrt{4 + \sin(0.05)}$.

Solution. (a) We have

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left[\sqrt{4+\sin(x)}\right] = \frac{d}{dx}\left[(4+\sin(x))^{1/2}\right] = \frac{1}{2}(4+\sin(x))^{-1/2}\frac{d}{dx}\left[4+\sin(x)\right] \\ &= \frac{1}{2}(4+\sin(x))^{-1/2}\cdot(0+\cos(x)) = \frac{\cos(x)}{2\sqrt{4+\sin(x)}}. \end{aligned}$$

We make the following table:

$$f(x) = \sqrt{4 + \sin(x)} \qquad f'(x) = \frac{\cos(x)}{2\sqrt{4+\sin(x)}}$$

$$a = 0$$

$$f(a) = \sqrt{4 + \sin(0)} = \sqrt{4} = 2 \qquad f'(a) = \frac{\cos(0)}{2\sqrt{4+\sin(0)}} = \frac{1}{2 \times 2} = \frac{1}{4}$$

So the microscope equation reads

$$\begin{aligned} f(a + \Delta x) &\approx f(a) + f'(a)\Delta x \\ f(0 + \Delta x) &\approx 2 + \frac{1}{4}\Delta x \\ \sqrt{4 + \sin(\Delta x)} &\approx 2 + \frac{1}{4}\Delta x. \end{aligned}$$

(b) Putting $\Delta x = 0.05$ into our result from part (a) gives

$$\sqrt{4 + \sin(0.05)} \approx 2 + \frac{1}{4} \times 0.05 = 2.0125.$$

(Compare this with the "true" numerical value $\sqrt{4 + \sin(0.05)} = 2.012456....$)

Example 2.6.4. Suppose $f(t)$ and $g(t)$ are differentiable functions, with $f(2) = 3$, $g(2) = 4$, $f'(2) = 2$, and $g'(2) = -1$. Write down the microscope equation for $h(t) = f(t)/g(t)$ at $t = 2$.

Solution. We have $h(2) = f(2)/g(2) = 3/4$, and

$$h'(2) = \frac{g(2)f'(2) - f(2)g'(2)}{(g(2))^2} = \frac{4 \times 2 - 3 \times (-1)}{4^2} = \frac{11}{16}.$$

So by the microscope equation,

$$h(t) \approx h(2) + h'(2)\Delta t = \frac{3}{4} + \frac{11}{16}\Delta t.$$

The microscope equation can also sometimes provide us with a heuristic, or "big picture," sense of how things change (approximately), as the following example illustrates.

Example 2.6.5. Show that, if the radius of a sphere increases by a small amount, then the volume of that sphere increase by approximately that same amount times the original surface area of the sphere. Use the fact that, again, a sphere of radius r has volume $V(r) = \frac{4}{3}\pi r^3$.

Solution. Let's imagine that our sphere initially has radius $r = a$, and that this radius expands a little bit, to $r = a + \Delta r$. By the microscope equation, its new volume is given approximately by:

$$V(a + \Delta r) \approx V(a) + V'(a)\Delta r = \frac{4}{3}\pi a^3 + 4\pi a^2 \Delta r.$$

That is, the new volume is roughly the original volume, $\frac{4}{3}\pi a^3$, plus the increase in radius Δr times the original surface area $4\pi a^2$. (This *is* the surface area of a sphere of radius a.)

The above result reflects our intuition that, because the added volume comprises a thin spherical "shell," this added volume is roughly equal to the surface area of that shell, times its thickness. Of course the two are not exactly equal. The shell has an inner surface *and* an outer surface, whose surface areas are different; because the shell is "curved," one cannot compute its volume simply by multiplying the inner surface area (or the outer) by the shell's thickness. But, as the above example shows, one can *approximate* this volume in this manner, and the thinner the shell, the more accurate the approximation.

Exercises

Part 1: The microscope equation for functions defined by simple formulas

1. Approximate $\sqrt[3]{7.9}$. Use $f(x) = \sqrt[3]{x} = x^{1/3}$, so that

$$f'(x) = \frac{1}{3}x^{-2/3} = \frac{1}{3(\sqrt[3]{x})^2}.$$

Also use $a = 8$.

2. Approximate $\sqrt[4]{620}$. Use $f(x) = \sqrt[4]{x}$. Hint: $\sqrt[4]{625} = 5$. How far is your estimate from the value given by a calculator?

3. (a) Write down the microscope equation for $y = \sqrt{x}$ at $x = 3600$.

(b) Use the microscope equation to estimate $\sqrt{3628}$ and $\sqrt{3592}$. How far are these estimates from the values given by a calculator?

4. (a) Write down the microscope equation for $h(x) = x^{-1/2} = 1/\sqrt{x}$ at $x = 25$.

(b) What estimate does the microscope equation give for $f(25.05) = 1/\sqrt{25.05}$? Calculate the true value of $1/\sqrt{25.05}$ and compare the two values; how far is the microscope estimate from the true value?

(c) What estimate does the microscope equation give for $1/\sqrt{24.5}$? How far is this from the true value?

(d) What estimate does the microscope equation give for $1/\sqrt{26}$? How far is this from the true value?

5. (a) Write down the microscope equation for $y = \sin(x)$ at $x = 0$.

(b) Using the microscope equation, estimate the following values: $\sin .3$, $\sin .007$, $\sin(-.02)$. Check these values with a calculator. (Remember to set your calculator to radian mode!)

6. (a) Write down the microscope equation for $f(x) = \sin(x)$ at $x = \pi$.

(b) Use this result to estimate $\sin(3)$.

7. (a) Write down the microscope equation for $y = \tan(x)$ at $x = 0$.

(b) Estimate the following values: $\tan(.007)$, $\tan(.3)$, $\tan(-.02)$. Check these values with a calculator.

8. (a) Write down the microscope equation for $f(x) = 2^x$ at $x = 4$. Use the fact that $f'(x) = \ln(2)2^x$ and $\ln(2) \approx 0.69315$.

(b) Estimate $2^{3.05}$ in two different ways:

(i) Use your above microscope equation with $\Delta x = -0.95$.

(ii) First, use your above microscope equation to estimate $2^{4.05}$. Then use the fact that

$$2^{3.05} = 2^{4.05-1} = \frac{2^{4.05}}{2^1} = \frac{2^{4.05}}{2}.$$

(c) Compare your above two estimates to the true value of $2^{3.05}$. Which estimate is better? Why do you think this is?

Part 2: The microscope equation for functions defined incompletely

9. (a) Suppose $y = f(x)$ is a function for which $f(5) = 12$ and $f'(5) = .4$. Write down the microscope equation for f at $x = 5$.

(b) What would you see if you were to graph $f(x)$ over the domain $4.999 \le x \le 5.001$?

(c) What estimate does the microscope equation give for $f(5.3)$?

(d) What estimates does the microscope equation give for the following: $f(5.23)$, $f(4.9)$, $f(4.82)$, $f(9)$? Do you consider these estimates to be equally reliable?

10. (a) Suppose $z = g(t)$ is a function for which $g(-4) = 7$ and $g'(-4) = 3.5$. Write down the microscope equation for g at $t = -4$.

(b) Estimate $g(-4.2)$ and $g(-3.75)$.

(c) For what value of t near -4 would you estimate that $g(t) = 6$? For what value of t would you estimate $g(t) = 8.5$?

11. If $f(a) = b$, $f'(a) = -3$ and if k is small, which of the following is the estimate to $f(a + k)$ given by the microscope equation?

$$a + 3k,\ b + 3k,\ a + 3b,\ b - 3k,\ a - 3k,\ 3a - b,\ a^2 - 3b,\ f'(a + k)$$

12. Suppose a person has travelled D feet in t seconds. Then $D'(t)$ is the person's velocity at time t; $D'(t)$ has units of feet per second.

(a) Suppose $D(5) = 30$ feet and $D'(5) = 5$ feet/second. Estimate the following:

$$D(5.1) \qquad D(5.8) \qquad D(4.7)$$

(b) If $D(2.8) = 22$ feet, while $D(3.1) = 26$ feet, what would you estimate $D'(3)$ to be?

13. Fill in the blanks.

(a) If $f(3) = 2$ and $f'(3) = 4$, a reasonable estimate of $f(3.2)$ is ____.

(b) If $g(7) = 6$ and $g'(7) = .3$, a reasonable estimate of $g(6.6)$ is ____.

(c) If $h(1.6) = 1$, $h'(1.6) = -5$, a reasonable estimate of $h($____$)$ is 0.

(d) If $F(2) = 0$, $F'(2) = .4$, a reasonable estimate of $F($____$)$ is .15.

(e) If $G(0) = 2$ and $G'(0) =$ ____, a reasonable estimate of $G(.4)$ is 1.6.

(f) If $H(3) = -3$ and $H'(3) =$ ____, a reasonable estimate of $H(2.9)$ is -1.

14. Let $f(t)$ and $g(t)$ be as in Example 2.6.4 above. Write down the microscope equation for $k(t) = f(t)g(t)$ at $t = 2$, and use this result to estimate $k(2.05)$.

15. In manufacturing processes, the profit is usually a function of the number of units being produced, among other things. Suppose we are studying some small industrial company that produces n units in a week and makes a corresponding weekly profit of P. Assume $P = P(n)$.

(a) If $P(1000) = \$500$ and $P'(1000) = \$2$/unit, then

$$P(1002) \approx \underline{\qquad} \qquad P(995) \approx \underline{\qquad} \qquad P(\underline{\qquad}) \approx \$512$$

(b) If $P(2000) = \$3000$ and $P'(2000) = -\$5$/unit, then

$$P(2010) \approx \underline{\qquad} \qquad P(1992) \approx \underline{\qquad} \qquad P(\underline{\qquad}) \approx \$3100$$

(c) If $P(1234) = \$625$ and $P(1238) = \$634$, then what is an estimate for $P'(1236)$?

Part 3: The microscope equation and the chain rule

For these exercises, you should refer especially to Example 2.6.3.

16. (a) Write down the microscope equation for the function $f(x) = \dfrac{1}{\sqrt[3]{8-x}}$ at $x = 0$.

(b) Using this microscope equation, estimate $\dfrac{1}{\sqrt[3]{7.99}}$ and $\dfrac{1}{\sqrt[3]{8.01}}$. How close are these answers to the "true" values you get from a calculator?

17. (a) Write down the microscope equation for $y = \sqrt{2}\cos(x/4)$ at $x = \pi$. Simplify your result using the fact that $\cos(\pi/4) = \sin(\pi/4) = 1/\sqrt{2}$.

(b) Use the microscope equation to estimate the values of $\cos(1.01\pi/4)$ and $\cos(.985\pi/4)$. How close are these answers to the "true" values you get from a calculator?

18. (a) Write down the microscope equation for $w = \sqrt[3]{8 + 2\tan(x)}$ at $x = 0$. (Recall that $\sqrt[3]{8} = 2$.)

(b) Use the microscope equation to estimate the values of $\sqrt[3]{8 + 2\tan(0.03)}$ and $\sqrt[3]{8 + 2\tan(.3)}$. Which one of these values do you think is closer to the true value? Answer without a calculator, and please explain. (You can check your answer with a calculator if you would like.)