2.5 More differentiation rules

A mathematical process is said to *commute* with differentiation if it doesn't matter whether you differentiate before, or after, you perform that process.

We've seen, for example, that differentiation commutes with addition of functions, and with multiplication of functions by constants. That is: the sum rule tells us that the derivative of the sum is the sum of the derivatives; the constant multiple rule tells us that, if you multiply a function before you differentiate that function, then you get the same result as if you differentiate first, and then multiply. (Food for thought: which of the latter two sequences of processes corresponds to which side of the constant multiple rule?) On the other hand, differentiation *does not* commute with composition of functions: the derivative of a chain is not, generally, equal to the chain of the derivatives.

In this section, we explore some additional processes that, like the chain rule, *do not* commute with differentiation.

The product rule

To see right away that the derivative of the product of two functions does not, in general, equal the product of the two derivatives of those functions, let $f(x) = x^2$ and $g(x) = x^3$. Then the derivative of the product is

$$\frac{d}{dx}[f(x)g(x)] = \frac{d}{dx}[x^2 \cdot x^3] = \frac{d}{dx}[x^5] = 5x^4,$$

while the product of the derivatives equals

$$\frac{d}{dx}[f(x)] \cdot \frac{d}{dx}[g(x)] = \frac{d}{dx}[x^2] \cdot \frac{d}{dx}[x^3] = 2x \cdot 3x^2 = 6x^3.$$

And certainly $5x^4 \neq 6x^3$ (except in the special cases $x = 0$ and $x = 6/5$)!

To understand how products *do* behave, in terms of change in their factors, it's helpful to consider an example. Suppose that, in January, your local Mathematical Society has q members, and charges each of them p dollars in monthly dues. Then the Society's total dues revenue r for January is

$$r = p\ \frac{\text{dollars}}{\text{member}} \times q \text{ members} = pq \text{ dollars}.$$

Now suppose that, in February, the Society acquires an additional Δq members, but does not raise its dues. Then the Society's *gain* in revenue, for February as compared to January, will be

$$\Delta r = p\ \frac{\text{dollars}}{\text{member}} \times \Delta q \text{ members} = p\Delta q \text{ dollars}.$$

Alternatively suppose that, in February, membership numbers stay the same, but dues *increase* by Δp dollars per member. Then the Society's gain in revenue, for February as compared to January, will be

$$\Delta r = q \text{ members} \times \Delta p\ \frac{\text{dollars}}{\text{member}} = q\Delta p \text{ dollars}.$$

In either case, we note the following: *If either factor in the product changes by a certain amount, then the product changes by that amount times the other factor.*

What happen if both factors change simultaneously? It's not quite correct to add the above two Δr's; things are just a bit more complicated. But we can argue as follows, in terms of the above example. Suppose dues increase by Δp dollars per member *and* membership increases by Δq members. Then the *gain* in revenue can be broken down into two categories: (i) $(p+\Delta p)\Delta q$ additional dollars in revenue from the *new* members, since each of the Δq new members pays dues of $p+\Delta p$ dollars; **plus** (ii) $q\Delta p$ additional dollars in revenue from the *original members*, since each of the q original members pays an extra Δp dollars (compared to what they were paying before the rate increase). That is,

$$\Delta r = (p+\Delta p)\Delta q + q\Delta p. \tag{2.5.1}$$

Of course, we derived equation (2.5.1) in a very particular setting, but an analogous argument shows that this equation holds for any product $r = pq$.

Now with this general context in mind, let's divide both sides of equation (2.5.1) through by Δx; we get

$$\frac{\Delta r}{\Delta x} = (p+\Delta p)\frac{\Delta q}{\Delta x} + q\frac{\Delta p}{\Delta x}. \tag{2.5.2}$$

We let Δx approach zero. The average rates of change become instantaneous rates of change, so equation (2.5.2) yields

$$\frac{dr}{dx} = p\frac{dq}{dx} + q\frac{dp}{dx}$$

or, writing $p = f(x)$ and $q = g(x)$,

$$\boxed{\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)}$$

The product rule

In words: the derivative of a product equals the first factor times the derivative of the second, plus the second factor times the derivative of the first.

Example 2.5.1. (a) Differentiate $y = x^2\cos(x)$.

(b) Find $\frac{d}{dx}\left[\sin(x^5+4)\cos(2^x)\right]$.

(c) Find $\frac{d}{dz}\left[\tan(z^5 3^z)\right]$.

(d) Suppose the per capita daily energy consumption in a country is currently 800,000 BTU, and, due to energy conservation efforts, it is falling at the rate of 1,000 BTU per year. Suppose too that the population of the country is currently 200,000,000 people, and is rising at the rate of 1,000,000 people per year. Is the total daily energy consumption of this country rising or falling? By how much?

Solution. (a) We have

$$\frac{dy}{dx} = x^2\frac{d}{dx}[\cos(x)] + \cos(x)\frac{d}{dx}[x^2] = x^2(-\sin(x)) + \cos(x)\cdot 2x = -x^2\sin(x) + 2x\cos(x).$$

(b) By the product rule, followed by two applications of the chain rule,

$$\begin{aligned}\frac{d}{dx}\big[\sin(x^5+4)\cos(2^x)\big] &= \sin(x^5+4)\frac{d}{dx}\big[\cos(2^x)\big] + \cos(2^x)\frac{d}{dx}\big[\sin(x^5+4)\big]\\ &= \sin(x^5+4)(-\sin(2^x))\frac{d}{dx}[2^x] + \cos(2^x)\cos(x^5+4)\frac{d}{dx}[x^5+4]\\ &= \sin(x^5+4)(-\sin(2^x))\cdot\ln(2)2^x + \cos(2^x)\cos(x^5+4)\cdot 5x^4\\ &= -\ln(2)2^x\sin(2^x)\sin(x^5+4) + 5x^4\cos(2^x)\cos(x^5+4).\end{aligned}$$

(c) By the chain rule, followed by the product rule,

$$\begin{aligned}\frac{d}{dz}\big[\tan(z^5 3^z)\big] &= \sec^2(z^5 3^z)\frac{d}{dz}\big[z^5 3^z\big]\\ &= \sec^2(z^5 3^z)\left(z^5\frac{d}{dz}[3^z] + 3^z\frac{d}{dz}[z^5]\right)\\ &= \sec^2(z^5 3^z)\big(z^5\cdot\ln(3)3^z + 3^z\cdot 5z^4\big)\\ &= \sec^2(z^5 3^z)3^z\big(\ln(3)z^5 + 5z^4\big).\end{aligned}$$

(d) We can model this situation with three functions $C(t)$, $P(t)$, and $E(t)$:

$$\begin{aligned}C(t) &: \text{per capita consumption at time } t\\ P(t) &: \text{population at time } t\\ E(t) &: \text{total energy consumption at time } t\end{aligned}$$

We are interested in whether, and how, $E(t)$ is increasing or decreasing; that is, we are interested in $E'(t)$.

Now total energy consumption equals per capita consumption times the number of people in the population; that is,

$$E(t) = C(t)P(t).$$

We differentiate using the product rule, to find that

$$E'(t) = C(t)P'(t) + P(t)C'(t). \tag{2.5.3}$$

Now if $t = 0$ represents today, then we are given the two rates of change

$$\begin{aligned}C'(0) &= -1{,}000 = -10^3 \text{ BTU per person per year, and}\\ P'(0) &= 1{,}000{,}000 = 10^6 \text{ persons per year.}\end{aligned}$$

Similarly, we're given $C(0)$ and $P(0)$. So by equation (2.5.3),

$$\begin{aligned} E'(0) &= C(0)P'(0) + P(0)C'(0) \\ &= (8 \times 10^5) \cdot (10^6) + (2 \times 10^8) \cdot (-10^3) \\ &= (8 \times 10^{11}) - (2 \times 10^{11}) \\ &= 6 \times 10^{11} \text{ BTU per year.} \end{aligned} \tag{2.5.4}$$

So the total daily energy consumption is currently rising at the rate of 6×10^{11} BTU per year. Note what this means: the growth in the population more than offsets the efforts to conserve energy.

In (2.5.4), we omitted the units until the very end. But the units do check out: $C(t)$ represents *per capita* daily energy consumption, so its units are BTU/person. Therefore, the units for $C(0) \cdot P'(0)$ are

$$\frac{\text{BTU}}{\text{person}} \cdot \frac{\text{persons}}{\text{year}} = \frac{\text{BTU}}{\text{year}},$$

and, similarly, the units for $P(0)C'(0)$ are

$$\text{persons} \times \frac{\text{BTU/person}}{\text{year}} = \frac{\text{BTU}}{\text{year}}.$$

The quotient rule

As a prelude to differentiating general quotients $f(x)/g(x)$ of functions, we first consider the particularly simple case when $f(x) = 1$: that is, we consider derivatives of *reciprocals* $1/g(x)$.

Recall that the derivative of $1/x$ is -1 over the square of x. So, by the chain rule, the derivative of one over a *function* of x is minus one over the square of that function of x, **times** the derivative of that function of x. (See, for example, "Steps for using the chain rule, version 2," on page 94.)

In symbols,

$$\frac{d}{dx}\left[\frac{1}{g(x)}\right] = -\frac{1}{(g(x))^2} \cdot \frac{d}{dx}[g(x)]$$

or, more simply,

$$\boxed{\frac{d}{dx}\left[\frac{1}{g(x)}\right] = -\frac{g'(x)}{(g(x))^2}}$$

The reciprocal rule

Example 2.5.2. (a) Find: (i) $\dfrac{d}{dt}\left[\dfrac{1}{t^2+1}\right]$; (ii) $\dfrac{d}{dt}\left[\dfrac{1}{(t^2+1)(t^2+2)}\right]$.

(b) Use the reciprocal rule (and what you know about derivatives of cosines) to show that $\dfrac{d}{dx}[\sec(x)] = \sec(x)\tan(x)$.

Solution. (a)(i) By the reciprocal rule,

$$\frac{d}{dt}\left[\frac{1}{t^2+1}\right] = -\frac{d[t^2+1]/dt}{(t^2+1)^2} = -\frac{2t}{(t^2+1)^2}.$$

(ii) By the reciprocal rule and the product rule,

$$\begin{aligned}\frac{d}{dt}\left[\frac{1}{(t^2+1)(t^2+2)}\right] &= -\frac{\frac{d}{dt}[(t^2+1)(t^2+2)]}{(t^2+1)^2(t^2+2)^2} = -\frac{(t^2+1)\frac{d}{dt}[t^2+2]+(t^2+2)\frac{d}{dt}[t^2+1]}{(t^2+1)^2(t^2+2)^2}\\ &= -\frac{(t^2+1)\cdot 2t+(t^2+2)\cdot 2t}{(t^2+1)^2(t^2+2)^2} = -\frac{2t(t^2+1+t^2+2)}{(t^2+1)^2(t^2+2)^2} = -\frac{2t(2t^2+3)}{(t^2+1)^2(t^2+2)^2}.\end{aligned}$$

(b) By the definitions of $\sec(x)$ and $\tan(x)$, by the reciprocal rule, and by some algebra, we have

$$\frac{d}{dx}[\sec(x)] = \frac{d}{dx}\left[\frac{1}{\cos(x)}\right] = -\frac{-\sin(x)}{\cos^2(x)} = \frac{\sin(x)}{\cos^2(x)} = \frac{1}{\cos(x)}\cdot\frac{\sin(x)}{\cos(x)} = \sec(x)\tan(x).$$

In our derivation of the reciprocal rule above, we began by noting that $\frac{d}{dx}\left[\frac{1}{x}\right] = -\frac{1}{x^2}$. But this latter formula is just the case $p=-1$ of the power formula. So the reciprocal rule is itself a special case of the power formula, combined with the chain rule.

We now turn to differentiation of general quotients. The key is to note that a quotient *is* a kind of product: in general, $a/b = (1/b)\cdot a$. So, by the above product and reciprocal rules,

$$\begin{aligned}\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] &= \frac{d}{dx}\left[\frac{1}{g(x)}\cdot f(x)\right] = \frac{1}{g(x)}\frac{d}{dx}[f(x)] + f(x)\frac{d}{dx}\left[\frac{1}{g(x)}\right]\\ &= \frac{1}{g(x)}\cdot f'(x) + f(x)\cdot\left(-\frac{g'(x)}{(g(x))^2}\right) = \frac{f'(x)}{g(x)} - \frac{f(x)g'(x)}{(g(x))^2}.\end{aligned} \tag{2.5.5}$$

To clean this up, we multiply top and bottom of the first quotient on the right-hand side of equation (2.5.5) by $g(x)$, to get a common denominator. We get

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x)}{(g(x))^2} - \frac{f(x)g'(x)}{(g(x))^2}$$

or, finally,

$$\boldsymbol{\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x)-f(x)g'(x)}{(g(x))^2}}$$

The quotient rule

Example 2.5.3. (a) Find: (i) $\frac{d}{dx}\left[\frac{2^x}{2^x+1}\right]$; (ii) $\frac{d}{dx}\left[\frac{x\cos(x)}{\sin(x^2)}\right]$.

(b) Use the reciprocal rule (and what you know about derivatives of cosines and sines) to show that $\frac{d}{dx}[\cot(x)] = -\csc^2(x)$.

Solution. (a)(i) By the quotient rule,

$$\begin{aligned}\frac{d}{dx}\left[\frac{2^x}{2^x+1}\right] &= \frac{(2^x+1)\frac{d}{dx}[2^x] - 2^x\frac{d}{dx}[2^x+1]}{(2^x+1)^2} = \frac{(2^x+1)\cdot\ln(2)2^x - 2^x\cdot\ln(2)2^x}{(2^x+1)^2}\\ &= \frac{\ln(2)2^x(2^x+1-2^x)}{(2^x+1)^2} = \frac{\ln(2)2^x}{(2^x+1)^2}.\end{aligned}$$

(ii) By the quotient rule, the product rule, and the chain rule,

$$\begin{aligned}\frac{d}{dx}\left[\frac{x\cos(x)}{\sin(x^2)}\right] &= \frac{\sin(x^2)\frac{d}{dx}[x\cos(x)] - x\cos(x)\frac{d}{dx}[\sin(x^2)]}{\sin^2(x^2)}\\ &= \frac{\sin(x^2)\left(x\frac{d}{dx}[\cos(x)] + \cos(x)\frac{d}{dx}[x]\right) - x\cos(x)\cos(x^2)\frac{d}{dx}[x^2]}{\sin^2(x^2)}\\ &= \frac{\sin(x^2)\big(-x\sin(x)+\cos(x)\big) - 2x^2\cos(x)\cos(x^2)}{\sin^2(x^2)}\\ &= \frac{-x\sin(x)\sin(x^2)+\cos(x)\sin(x^2) - 2x^2\cos(x)\cos(x^2)}{\sin^2(x^2)}.\end{aligned}$$

(b) By the definitions of $\cot(x)$ and $\csc(x)$, the quotient rule, the trigonometric identity

$$\cos^2(x) + \sin^2(x) = 1,$$

and some algebra, we have

$$\begin{aligned}\frac{d}{dx}[\cot(x)] = \frac{d}{dx}\left[\frac{\cos(x)}{\sin(x)}\right] &= \frac{\sin(x)\frac{d}{dx}[\cos(x)] - \cos(x)\frac{d}{dx}[\sin(x)]}{\sin^2(x)}\\ &= \frac{\sin(x)(-\sin(x)) - \cos(x)\cos(x)}{\sin^2(x)} = \frac{-(\sin^2(x)+\cos^2(x))}{\sin^2(x)}\\ &= \frac{-1}{\sin^2(x)} = -\left(\frac{1}{\sin(x)}\right)^2 = -\csc^2(x).\end{aligned}$$

Summary of differentiation rules

Here are all the differentiation rules that we have encountered thus far.

function $y = h(x)$	derivative $\dfrac{dy}{dx} = h'(x) = \dfrac{d}{dx}[h(x)]$	name of rule
$cf(x)$	$cf'(x)$	constant multiple rule
$f(x) + g(x)$	$f'(x) + g'(x)$	sum rule
$f(x)g(x)$	$f(x)g'(x) + g(x)f'(x)$	product rule
$\dfrac{1}{g(x)}$	$-\dfrac{g'(x)}{(g(x))^2}$	reciprocal rule
$\dfrac{f(x)}{g(x)}$	$\dfrac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$	quotient rule

Table 2.2 A short table of differentiation rules

Here c is an arbitrary real number, and $f(x)$ and $g(x)$ are any differentiable functions. (The reciprocal and quotient rules hold only for those x such that $g(x) \neq 0$.)

The reciprocal rule may be considered a special case of the quotient rule (see the exercises below).

Exercises

Part 1: Finding Derivatives

1. Find the derivative of each of the following functions.

(a) $3x^5 - 10x^2 + 8$
(b) $(5x^{12} + 2)(\pi - \pi^2 x^4)$
(c) $\sqrt{u} - 3/u^3 + 2u^7$
(d) $mx + b$ (m, b constant)
(e) $.5\sin(x) + \sqrt[3]{x} + \pi^2$
(f) $\dfrac{\pi - \pi^2 x^4}{5x^{12} + 2}$
(g) $2\sqrt{x} - \dfrac{1}{\sqrt{x}}$
(h) $\tan(z)\,(\sin(z) - 5)$
(i) $\dfrac{\sin(x)}{x^2}$
(j) $x^2 3^x$
(k) $\cos(x) + 5^x$
(l) $\sin(x)/\cos(x)$
(m) $5^x \cos(x)$
(n) $\dfrac{2^x}{10 + \sin(x)}$
(o) $\sin(4^x \cos(x))$
(p) $64^{\cos(t)}/(5\sqrt[3]{t})$
(q) $4^{x^2 + x2^x}$
(r) $\dfrac{5x^2 + \cos(x)}{7\sqrt{x} + 5}$

2. Suppose f and g are functions and that we are given

$$f(2) = 3, \quad g(2) = 4, \quad g(3) = 2,$$
$$f'(2) = 2, \quad g'(2) = -1, \quad g'(3) = 17.$$

Evaluate the derivative of each of the following functions at $t = 2$:

(a) $f(t)+g(t)$
(b) $5f(t)-2g(t)$
(c) $f(t)g(t)$
(d) $\dfrac{f(t)}{g(t)}$
(e) $g(f(t))$
(f) $\sqrt{g(t)}$
(g) $t^2 f(t)$
(h) $(f(t))^2+(g(t))^2$
(i) $\dfrac{1}{f(t)}$
(j) $f(3t-(g(1+t))^2)$

(k) What additional piece of information would you need to calculate the derivative of $f(g(t))$ at $t=2$?

3. (a) Use the product rule twice to show that

$$\frac{d}{dx}[f(x)g(x)h(x)] = f(x)g(x)h'(x) + f(x)h(x)g'(x) + g(x)h(x)f'(x).$$

(b) If the length, width, and height of a rectangular box are changing at the rates of 3, 6, and -5 inches/minute at the moment when all three dimensions happen to be 10 inches, at what rate is the volume of the box changing then?

(c) If the length, width, and height of a box are 10 inches, 12 inches, and 8 inches, respectively, and if the length and height of the box are changing at the rates of 3 inches/minute and -2 inches/minute, respectively, at what rate must the width be changing to keep the volume of the box constant?

4. Which of the following functions has a derivative which is always positive (except at $x=0$, where neither the function nor its derivative is defined)? Please explain your answer.

$$1/x \qquad -1/x \qquad 1/x^2 \qquad -1/x^2$$

5. Do the following.

(a) Show that $\dfrac{1}{1-x^2}$ and $\dfrac{x^2}{1-x^2}$ have the same derivative.

(b) If $f'(x)=g'(x)$ for every x, what can be concluded about the relationship between f and g? (Hint: What is $\dfrac{d}{dx}[f(x)-g(x)]$?)

(c) Show that $\dfrac{1}{1-x^2} = \dfrac{x^2}{1-x^2} + C$ by finding C.

6. Suppose that the current total daily energy consumption in a particular country is 16×10^{13} BTU and is rising at the rate of 6×10^{11} BTU per year. Suppose that the current population is 2×10^8 people and is rising at the rate of 10^6 people per year. What is the current daily per capita energy consumption? Is it rising or falling? By how much?

7. The population of a particular country is 15,000,000 people and is growing at the rate of 10,000 people per year. In the same country the per capita yearly expenditure for energy is $1,000

per person and is growing at the rate of \$8 per year. What is the country's current total yearly energy expenditure? How fast is the country's total yearly energy expenditure growing?

8. The population of a particular country is 30 million and is rising at the rate of 4,000 people per year. The total yearly personal income in the country is 20 billion dollars, and it is rising at the rate of 500 million dollars per year. What is the current per capita personal income? Is it rising or falling? By how much?

Part 2: Deriving Differentiation Rules

9. In this problem we calculate the derivative of $f(x) = x^4$.

(a) Expand $f(x + \Delta x) = (x + \Delta x)^4 = (x + \Delta x)(x + \Delta x)(x + \Delta x)(x + \Delta x)$ as a sum of 16 terms. (Don't collect "like" terms yet.)

(b) How many terms in part a involve *no* Δx's? What form do such terms have?

(c) How many terms in part a involve exactly *one* Δx? What form do such terms have?

(d) Group the terms in part a so that $f(x + \Delta x)$ has the form

$$Ax^4 + B\Delta x + R(\Delta x)^2 \ ,$$

where there are no Δx's among the terms in A or B, but R has several terms, some involving Δx. Use part b to check your value of A; use part c to check your value of B.

(e) Compute the quotient $\dfrac{f(x + \Delta x) - f(x)}{\Delta x}$, taking advantage of part d.

(f) Now find

$$\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x};$$

this is the derivative of x^4. Is your result here compatible with the rule for the derivative of x^n ?

10. In this problem we calculate the derivative of $f(x) = x^n$, where n is any positive integer.

(a) First show that you can write

$$f(x + \Delta x) = x^n + nx^{n-1}\Delta x + R(\Delta x)^2$$

by developing the following line of argument. Write $(x + \Delta x)^n$ as a product of n identical factors:

$$(x + \Delta x)^n = \underbrace{(x + \Delta x)}_{\text{1st}}\underbrace{(x + \Delta x)}_{\text{2nd}}\underbrace{(x + \Delta x)}_{\text{3rd}}\ldots\underbrace{(x + \Delta x)}_{n\text{th}}$$

But now, before tackling this general case, look at the following examples. In the examples we use notation to help us keep track of which factors are contributing to the final result.

i) Consider the product $(a+b)(\underline{a}+\underline{b}) = a\underline{a} + a\underline{b} + b\underline{a} + b\underline{b}$. There are four individual terms. Each term contains one of the entries in the first factor (namely a or b) and one of the entries in the second factor (namely $\underline{a}$ or $\underline{b}$). The four terms represent thereby all possible ways of choosing one entry in the first factor and one entry in the second factor.

ii) Multiply out the product $(a+b)(\underline{a}+\underline{b})(A+B)$. (Don't combine like terms yet.) Does each term contain one entry from the first factor, one from the second, and one from the third? How many terms did you get? In fact there are two ways to choose an entry from the first factor, two ways to choose an entry from the second factor, and two ways to choose an entry from the third factor. Therefore, how many ways can you make a choice consisting of one entry from the first, one from the second, and one from the third?

Now return to the general case:

$$(x+\Delta x)^n = \underbrace{(x+\Delta x)}_{\text{1st}}\underbrace{(x+\Delta x)}_{\text{2nd}}\underbrace{(x+\Delta x)}_{\text{3rd}}\cdots\underbrace{(x+\Delta x)}_{n\text{th}}$$

How many ways can you choose an entry from each factor and *not* get any Δx's? Multiply these chosen entries together; what does the product look like (apart from having no Δx's in it)?

How many ways can you choose an entry from each factor in such a way that the resulting product has *precisely one* Δx? Describe all the various choices which give that result. What does a product that contains precisely one Δx factor look like? What do you obtain for the sum of *all* such terms with precisely one Δx factor?

What is the minimum number of Δx factors in any of the remaining terms in the full expansion of $(x+\Delta x)^n$?

Do your calculations agree with this summary:

$$(x+\Delta x)^n = x^n + nx^{n-1}\Delta x + R(\Delta x)^2 \text{ ?}$$

(b) Now find the value of $\dfrac{f(x+\Delta x)-f(x)}{\Delta x}$.

(c) Finally, find

$$\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}.$$

Do you get nx^{n-1}?

11. In this exercise we prove the sum rule: $F(x) = f(x)+g(x)$ implies $F'(x) = f'(x)+g'(x)$.

(a) Show $F(x+\Delta x) - F(x) = f(x+\Delta x) - f(x) + g(x+\Delta x) - g(x)$.

(b) Divide by Δx and finish the argument.

12. Recall that the quotient rule says: if $F(x) = f(x)/g(x)$, then

$$F'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}.$$

In this exercise, we prove the quotient rule in a different way.

(a) Rewrite $F(x) = f(x)/g(x)$ as $f(x) = g(x)F(x)$. Pretend for the moment that you know what $F'(x)$ is and apply the Product Rule to find $f'(x)$ in terms of $F(x)$, $g(x)$, $F'(x)$, $g'(x)$.

(b) Replace $F(x)$ by $f(x)/g(x)$ in your expression for $f'(x)$ in part a.

(c) Solve the equation in part b for $F'(x)$ in terms of $f(x)$, $g(x)$, $f'(x)$ and $g'(x)$.

13. In this problem we calculate the derivative of $f(x) = x^n$ when n is a negative integer. First write $n = -m$, so m is a positive integer. Then $f(x) = x^{-m} = 1/x^m$.

(a) Use the Quotient Rule and this new expression for f to find $f'(x)$.

(b) Do the algebra to re-express $f'(x)$ as nx^{n-1}.

14. In this problem we calculate the derivatives of $\sin(x)$ and $\cos(x)$. We will need the **addition formulas**:

$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$
$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

First tackle $f(x) = \sin(x)$:

(a) Use the addition formula for $\sin(A + B)$ to rewrite $f(x + \Delta x)$ in terms of $\sin(x)$, $\cos(x)$, $\sin(\Delta x)$, and $\cos(\Delta x)$.

(b) The quotient $\dfrac{f(x+\Delta x) - f(x)}{\Delta x}$ can now be written in the form

$$P(\Delta x) \cdot \sin(x) + Q(\Delta x) \cdot \cos(x)\,,$$

where P and Q are specific functions of Δx. What are the formulas for those functions?

(c) Use a calculator or computer to estimate the limits

$$\lim_{\Delta x \to 0} P(\Delta x) \qquad \text{and} \qquad \lim_{\Delta x \to 0} Q(\Delta x)\ .$$

(Try $\Delta x = .1, .01, .001, .0001$. Be sure your calculator is set on radians, not degrees.) Using part b you should now be able to determine the limit

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

by writing it in the form

$$\left(\lim_{\Delta x \to 0} P(\Delta x)\right) \cdot \sin(x) + \left(\lim_{\Delta x \to 0} Q(\Delta x)\right) \cdot \cos(x)\ .$$

(d) What is $f'(x)$?

(e) Proceed similarly to find the derivative of $g(x) = \cos(x)$.

15. In this problem we calculate the derivatives of the other circular functions. Use the quotient rule together with the derivatives of $\sin(x)$ and $\cos(x)$ to verify that the derivatives of the other four circular functions are as given in the table below:

function	derivative
$\tan x = \dfrac{\sin(x)}{\cos(x)}$	$\sec^2 x$
$\csc x = \dfrac{1}{\sin(x)}$	$-\cot x \csc x$
$\sec x = \dfrac{1}{\cos(x)}$	$\sec x \tan x$
$\cot x = \dfrac{1}{\tan x}$	$-\csc^2 x$

Part 3: Second derivatives

If $y = f(x)$, then the **second derivative** of f is just the derivative of the derivative of f; it is denoted $f''(x)$ or d^2y/dx^2. For example, the second derivative of $f(x) = x^7$ is found by first computing that $f'(x) = 7x^6$, so that $f''(x) = d[7x^6]/dx = 7 \cdot 6x^5 = 42x^5$.

16. Find the second derivative of each of the following functions.

(a) $f(x) = 4x^3 - 7x^2 - 15x + 11$

(b) $f(t) = 2^{3t-2}$

(c) $f(x) = \sin \omega x$, where ω is a constant

(d) $f(x) = x^2 \cos(x)$

17. Show that $\sin \omega x$ satisfies the differential equation $y'' + \omega^2 y = 0$. What other solutions can you find to this differential equation? Can you find a function $L(x)$ that satisfies these three conditions:

$$\begin{aligned} L''(x) + 4L(x) &= 0 \\ L(0) &= 36 \\ L'(0) &= 64? \end{aligned}$$

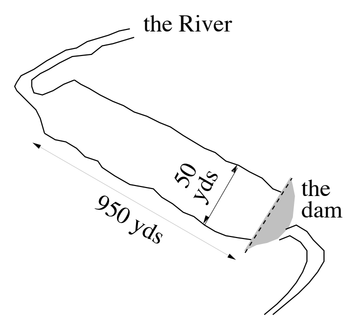
Part 4: the Colorado River problem

.
Make your answer to this sequence of questions an essay. Identify all the variables you consider (e.g., "A stands for the area of the lake"), and indicate the functional relationships between them

("A depends on time t, measured in weeks from the present"). Identify the derivatives of those functions, as necessary.

The Colorado River—which excavated the Grand Canyon, among others—used to empty into the Gulf of California. It no longer does. Instead, it runs into a marshy area some miles from the Gulf and stops. One of the major reasons for this change is the construction of dams—notably the Hoover Dam. Every dam creates a lake behind it, and every lake increases the total surface area of the river. Since the rate at which water evaporates is proportional to the area of the water surface exposed to air, the lakes along the Colorado have increased the loss of river water through evaporation. Over the years, these losses (in conjunction with other factors, like increased usage by a rapidly growing population) have been significant enough to dry up the river at its mouth.

18. Let us analyze the evaporation rate along a river that was recently dammed. Suppose the lake is currently 50 yards wide, and getting wider at a rate of 3 yards per week. As the lake fills, it gets longer, too. Suppose it is currently 950 yards long, and it is extending upstream at a rate of 15 yards per week. Assuming the lake remains approximately rectangular as it grows, find



(a) the current area of the lake, in square yards;

(b) the rate at which the surface of the lake is currently growing, in square yards per week.

19. Suppose the lake continues to spread sideways at the rate of 3 yards per week, and it continues to extend upstream at the rate of 15 yards per week.

(a) Express the area of the lake as a (quadratic!) function of time, where time is measured from the present, in weeks, and where the lake's area is as given in Exercise 20.

(b) How many weeks will it take for the lake to cover 30 acres (= 145,200 square yards)?

(c) At what rate is the lake surface growing when it covers 30 acres?

20. Compare the rates at which the surface of the lake is growing in Exercise 19 (which is the "current" rate) and in Exercise 20(c) (which is the rate when the lake covers 30 acres). Are these rates the same? If they are not, how do you account for the difference? In particular, the width and length grow at fixed rates, so why doesn't the area? Use what you know about derivatives to answer the question.

21. Suppose the local climate causes water to evaporate from the surface of the lake at the rate of 0.22 cubic yards per week, for each square yard of surface. Write a formula that expresses total evaporation per week in terms of area. Use E to denote total evaporation.

22. The lake is fed by the river, and that in turn is fed by rainwater and groundwater from its watershed. (The **watershed**, or basin, of a river is that part of the countryside containing

the ponds and streams which drain into the river.) Suppose the watershed provides the lake, on average, with 25,000 cubic yards of new water each week.

Assuming, as we did in Exercise 19, that the lake widens at the constant rate of 3 yards per week, and lengthens at the rate of 15 yards per week, will the time ever come that the water being added to the lake from its watershed balances the water being removed by evaporation? In other words, will the lake ever stop filling?