Chapter 2

The derivative

In studying SIR and other phenomena in Chapter 1, we contented ourselves with an intuitive, or heuristic, understanding of what a rate of change actually *is*. That is, we never defined **rate of change** in a mathematically precise way.

In this chapter, we will provide such a definition. Two such definitions, actually – one of an *average* rate of change, also known as a **difference quotient**, and one of the *instantaneous* rate of change, also known as the **derivative**.

We've encountered both notions in the previous chapter. We now investigate these ideas more formally, and in greater depth.

2.1 Rates of change

By an *average rate of change* of an output with respect to an input, we mean the net change in output divided by the corresponding change in input. In preceding discussions, we have generally taken our input variable to be time t, but other independent variables are possible, as illustrated in the following example.

Example 2.1.1. Water density. Under appropriate atmospheric conditions, the density of water, as a function of water temperature, may be modeled fairly well by the formula

$$D(C) = 999.973 - 0.008(C - 4.06)^2,$$

where C is temperature in degrees Celsius (°C) and D is density in kilograms per cubic meter (kg/m^3). This formula holds reasonably well for C between about 0 and 8 degrees Celsius.

(a) Find the average rate of change of D with respect to C, over each of the following intervals (of temperature values, in °C): [1,6], [1,3], [1,2], [1,1.5], [1,1.1], and [1,1.01]. What are the appropriate units for these rates of change?

(b) Repeat part (a), but this time with these temperature intervals: [2,7], [2,4], [2,3], [2,2.5], [2,2.1], and [2,2.01].

(c) Can we make sense of "the *instantaneous* rate of change of water density with respect to temperature, *at* 1 degree Celsius"? If so, what numerical value might we give this instantaneous rate of change? Answer the same questions for $C = 2$ degrees Celsius.

Solution (a) Over the interval [1,6], the average rate of change is

$$\frac{\Delta D}{\Delta C} = \frac{D(6) - D(1)}{6-1}\,\frac{\text{kg/m}^3}{^\circ\text{C}} = \frac{999.973 - 0.008(6-4.06)^2 - (999.973 - 0.008(1-4.06)^2)}{5}\,\frac{\text{kg/m}^3}{^\circ\text{C}}$$
$$= \frac{999.943 - 999.898}{5}\,\frac{\text{kg/m}^3}{^\circ\text{C}} = 0.009\,\frac{\text{kg/m}^3}{^\circ\text{C}}.$$

Over [1,3], the average rate of change is

$$\frac{\Delta D}{\Delta C} = \frac{D(3) - D(1)}{3-1}\,\frac{\text{kg/m}^3}{^\circ\text{C}} = \frac{999.964 - 999.898}{2}\,\frac{\text{kg/m}^3}{^\circ\text{C}} = 0.033\,\frac{\text{kg/m}^3}{^\circ\text{C}}.$$

In a similar manner, we find the remaining entries of the following table. (All entries in the second row are in $(\text{kg/m}^3)/^\circ\text{C}$.)

Interval	[1,6]	[1,3]	[1,2]	[1,1.5]	[1,1.1]	[1,1.01]
$\Delta D/\Delta C$	0.009	0.033	0.041	0.045	0.048	0.049

(b) We need to compute

$$\frac{\Delta D}{\Delta C} = \frac{D(x) - D(2)}{x-2}$$

for various values of x, getting closer and closer to 2 (namely, $x = 7, 4, 3, 2.5, 2.1, 2.01$). The computations are much as in part (a), and are summarized in the table below. (Again, all average rates of change are in $(\text{kg/m}^3)/^\circ\text{C}$.)

Interval	[2,7]	[2,4]	[2,3]	[2,2.5]	[2,2.1]	[2,2.01]
$\Delta D/\Delta C$	-0.007	0.017	0.025	0.029	0.032	0.033

(c) In part (a), we computed the average rate of change $\Delta D/\Delta C$ over shorter and shorter temperature intervals $[1{,}1 + \Delta C]$. We might think of the *instantaneous* rate of change of D with respect to C, at $C = 1$, as "what happens to these average rates of change as the intervals $[1{,}1 + \Delta C]$ become *infinitesimally* short." Now observe from our computations in part (a) that, the shorter our interval $[1{,}1 + \Delta C]$ – that is, the smaller our ΔC – the more $\Delta D/\Delta C$ appears to zero in on 0.049. So we might say that the instantaneous rate of change of D with respect to C, at $C = 1$, is about $0.049\,(\text{kg/m}^3)/^\circ\text{C}$.

Similarly, according to our computations in part (b), we might say that the instantaneous rate of change of D with respect to C, at $C = 2$, is about $0.033\,(\text{kg/m}^3)/^\circ\text{C}$.

As the above example indicates, average rates of change may be expressed mathematically in terms of functions.

Definition 2.1.1. Consider a function $y = f(x)$. Suppose x changes from a point $x = a$ to a point $x = a + \Delta x$ (so that x changes by Δx). Then the corresponding change in y is

$$\Delta y = f(a + \Delta x) - f(a),$$

and we define the **average rate of change of f, or of y, from $x = a$ to $x = a + \Delta x$**, to be the **difference quotient**

$$\frac{\Delta y}{\Delta x} = \frac{f(a + \Delta x) - f(a)}{\Delta x}. \tag{2.1.1}$$

Part (c) of the above example points to a crucial idea: the interpretation of an instantaneous rate of change as a *limit* of average rates of change, as we average over shorter and shorter intervals. That is: suppose we can somehow ascribe an actual mathematical value to "what happens to the average rate of change (2.1.1) as Δx shrinks to zero." Then we should call this value "the instantaneous rate of change of $f(x)$ at $x = a$."

Another name for such an instantaneous rate of change is **derivative.** The formal definition is as follows.

Definition 2.1.2. Given a function $y = f(x)$ and a point $x = a$, we define the **instantaneous rate of change**, or **derivative, of $y = f(x)$ at $x = a$**, denoted $f'(a)$, to be "what happens to the average rate of change (2.1.1) as Δx shrinks to zero." In symbols,

$$f'(a) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(a + \Delta x) - f(a)}{\Delta x}, \tag{2.1.2}$$

where the notation "$\lim_{\Delta x \to 0}$" is pronounced "the limit, as Δx approaches zero." This definition applies whenever the limit in question exists.

The above definition is only "formal" insofar as the notion of "limit" is formal. We will content ourselves with a *working* notion of "limit" – a notion that will allow us to *compute* some derivatives, but will also provide some insight into how and when a derivative might *fail* to exist. We'll return to the latter issue in the next section. In the meantime, here are some computations that work.

Example 2.1.2. Let $f(x) = x^2$. Find:

(a) The average rate of change of $f(x)$ with respect to x, from $x = 3$ to $x = 3.1$, and from $x = 3$ to $x = 3.01$;

(b) The average rate of change of $f(x)$ from $x = 3$ to $x = 3 + \Delta x$, for an arbitrary $\Delta x \neq 0$;

(c) The instantaneous rate of change of $f(x)$ at $x = 3$.

Solution. (a) For the first of the two average rates of change, we set $a = 3$ and $\Delta x = 0.1$. Then

$$\frac{\Delta y}{\Delta x} = \frac{f(a + \Delta x) - f(a)}{\Delta x} = \frac{f(3.1) - f(3)}{0.1} = \frac{3.1^2 - 3^2}{0.1} = \frac{9.61 - 9}{0.1} = \frac{0.61}{0.1} = 6.1.$$

Similarly, for $\Delta x = 0.01$, we have

$$\frac{\Delta y}{\Delta x} = \frac{f(3.01) - f(3)}{0.01} = \frac{3.01^2 - 3^2}{0.01} = \frac{9.0601 - 9}{0.01} = \frac{0.0601}{0.1} = 6.01.$$

(b) Here, we find that

$$\begin{aligned} \frac{\Delta y}{\Delta x} &= \frac{f(a+\Delta x) - f(a)}{\Delta x} = \frac{f(3+\Delta x) - f(3)}{\Delta x} = \frac{(3+\Delta x)^2 - 3^2}{\Delta x} \\ &= \frac{9 + 6\Delta x + (\Delta x)^2 - 9}{\Delta x} = \frac{6\Delta x + (\Delta x)^2}{\Delta x} = \frac{\Delta x(6+\Delta x)}{\Delta x} = 6 + \Delta x. \end{aligned} \tag{2.1.3}$$

(c) It's quite clear that the the limit, as Δx approaches zero, of the right-hand side of (2.1.3) equals 6. But the left-hand and right-hand sides of (2.1.3) are equal, so the limit of the right-hand side must equal the limit of the left-hand side. And the limit of the left-hand side *is*, by Definition 2.1.2, the instantaneous rate of change of $f(x)$ at $x = 3$, also denoted $f'(3)$. In sum,

$$f'(3) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} (6 + \Delta x) = 6.$$

Note that the evaluation of $f'(3)$, in part (c) of the previous example, relied heavily on the *algebra* employed in part (b). Specifically, in part (b) we were able to simplify the *numerator* Δy of our difference quotient, to the point where we could factor Δx out of this numerator. We then *cancelled* this factor against the Δx in the denominator. This was crucial because, had a factor of Δx *remained* in the denominator, then letting $\Delta x \to 0$ in part (c) would have effectively left us with a *zero* in the denominator, and we know that can't be good!

Derivative computations will typically entail some type of "cancellation in numerator and denominator." However, that cancellation can take a variety of forms, one of which is illustrated in the following example.

Example 2.1.3. Let $h(x) = \sin(x)$. Find $h'(0)$. Use the "trigonometric limit formula"

$$\lim_{\theta \to 0} \frac{\sin(\theta)}{\theta} = 1. \tag{2.1.4}$$

(This formula is explored in the exercises below. Heuristically it says that, as an angle shrinks to zero, its sine and its radian measure become very close to each other.)

Solution. By Definition 2.1.2 of the derivative, we have

$$h'(0) = \lim_{\Delta x \to 0} \frac{h(0+\Delta x) - h(0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\sin(\Delta x) - \sin(0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\sin(\Delta x) - 0}{\Delta x} = \lim_{\Delta x \to 0} \frac{\sin(\Delta x)}{\Delta x} = 1,$$

the last step by (2.1.4) (with $\theta = \Delta x$).

We will explore other derivative computations in the exercises below. In particular, we'll use the definition of the derivative to confirm our intuition about instantaneous rates of change in the water density context (Example 2.1.1) above.

We now wish to interpret the derivative geometrically. To do this, let's note that the average rate of change

$$\frac{\Delta y}{\Delta x} = \frac{f(a + \Delta x) - f(a)}{\Delta x}$$

is just the slope of the line through the points $(a,f(a))$ and $(a + \Delta x, f(a + \Delta x))$ on the graph of f. This line is called a *secant line* to the graph of $f(x)$, meaning a line that intersects this graph in (at least) two points.

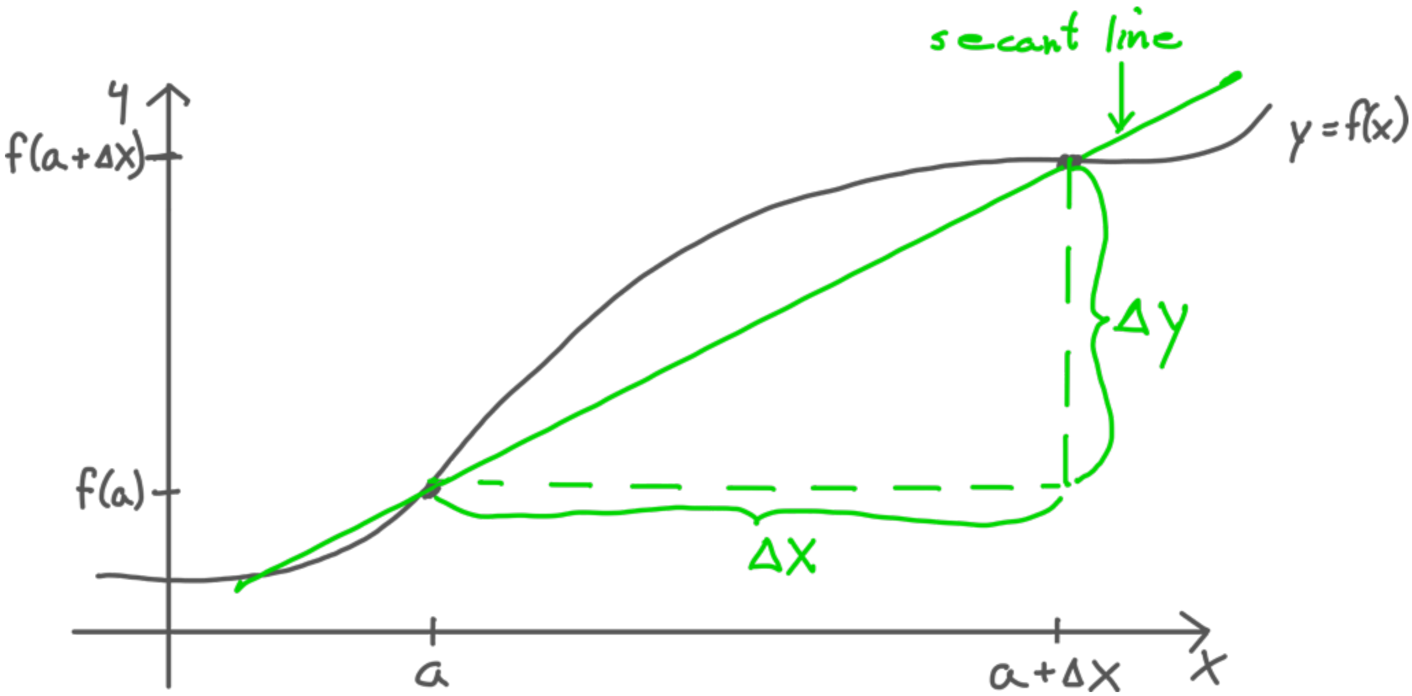

Figure 2.1. An average rate of change is the slope of a secant line

What does this have to do with derivatives? Well: note that, by the definition of the derivative and by the above geometric arguments,

$$\begin{aligned} f'(a) &= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(a + \Delta x) - f(a)}{\Delta x} \\ &= \lim_{\Delta x \to 0} [\text{slope of the secant line through } (a,f(a)) \text{ and } (a + \Delta x, f(a + \Delta x))]. \end{aligned} \tag{2.1.5}$$

But as $\Delta x \to 0$, the secant lines in question approach the *tangent line* to the graph of $f(x)$ at the point $x = a$. See Figure 2.2 below. (The tangent line is the line "just touching" the graph of $f(x)$ at the point in question. Intuitively, one can think of the tangent line as the "secant line through $(a,f(a))$ and $(a + \Delta x, f(a + \Delta x))$, where Δx is infinitesimally small.") So the *slopes* of these secant lines approach the *slope* of the tangent line, as $\Delta x \to 0$. Or in other words, the slope of this tangent line is the *limit*, as $\Delta x \to 0$, of the slopes of these secant lines. So by (2.1.5), the slope of this tangent line is $f'(a)$!

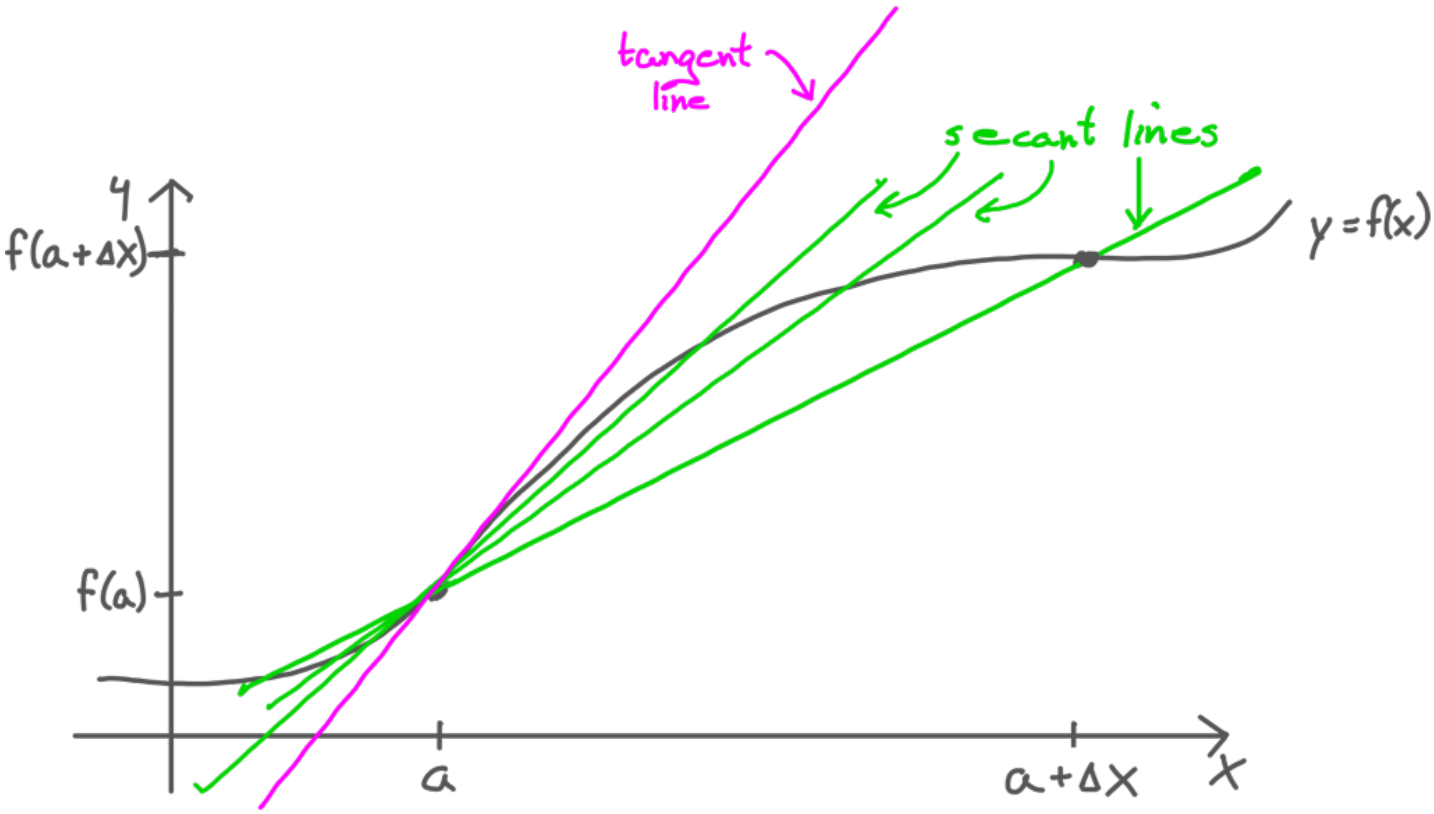

Figure 2.2. As Δx shrinks, the secant lines become the tangent line

We summarize:

> **The instantaneous rate of change, or derivative, $f'(a)$, equals the slope of the line tangent to the graph of $y = f(x)$ at the point $x = a$.**

The derivative at a point is the slope of the tangent line at that point

Example 2.1.4. Find the equation of the line tangent to $f(x) = x^2$ at $x = 3$.

Solution. This line passes through the point $(3,f(3)) = (3,3^3) = (3,9)$, and has slope equal to the derivative $f'(3)$ of $f(x)$ at $x = 3$. We saw in Example 2.1.2 that $f'(3) = 6$. So, by the initial-value formula (1.5.1), the tangent line has equation

$$y = f(3) + f'(3)(x-3) = 3^2 + 6(x-3) = 9 + 6x - 18 = 6x - 9.$$

Part (d) of the above example illustrates a general fact. Suppose we have an arbitrary function $f(x)$, and an arbitrary point $x = a$, and suppose $f'(a)$ exists. Then the line tangent to the graph of f has slope $f'(a)$, and passes through the point $(a,f(a))$. Consequently, by the initial-value formula (1.5.1), this tangent line has equation

$$y = f(a) + f'(a)(x - a). \tag{2.1.6}$$

Equation of the line tangent to $y = f(x)$ at $x = a$

Again, this formula makes sense only in situations where $f'(a)$ exists.

For instance, Example 2.1.3 above tells us that the tangent line to the graph of $h(0) = \sin(x)$ at $x = 0$ has equation

$$y = h(0) + h'(0)(x - 0) = 0 + 0x = x.$$

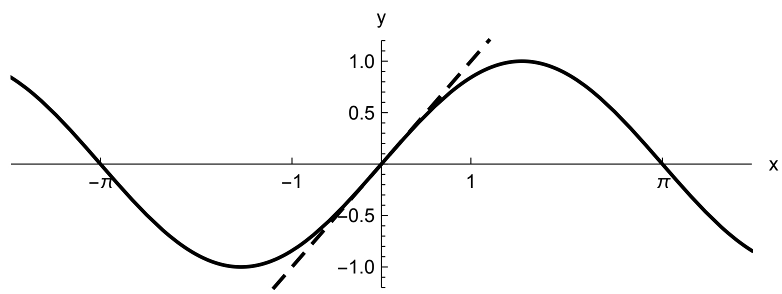

Figure 2.3. **The graph of $y = \sin(x)$ and its tangent line at $x = 0$**

To summarize the geometry of rates of change: an average rate of change is the slope of a *secant* line; an instantaneous rate of change is the slope of a *tangent* line.

We will often refer to "the slope of $y = f(x)$ at $x = a$" when we mean "the slope of the line *tangent* to $y = f(x)$ at $x = a$." Again, this slope is just $f'(a)$ (when $f'(a)$ exists). So we think of the derivative of a function, at a given point, as telling us the slope of that function at that point.

Exercises

1. Let $f(x) = 2x^2 - 3$.

(a) Find the average rate of change $\Delta y/\Delta x$ of $f(x)$ with respect to x, from $x = 2$ to $x = 2 + \Delta x$, for each of the following three values of Δx: $\Delta x = 0.1$, $\Delta x = 0.01$, $\Delta x = 0.001$.

(b) Based on part (a) above, what might you guess $f'(2)$ is equal to?

(c) Use algebra to show that the average rate of change of $f(x)$ with respect to x, from $x = 2$ to $x = 2 + \Delta x$, is $8 + 2\Delta x$.

(d) Find the instantaneous rate of change of $f(x)$ at $x = 2$.

(e) Find the equation of the line tangent to the graph of $f(x)$ at $x = 2$.

2. Repeat Exercise 1 with the same function $f(x)$, but this time, at $x = -1$. (That is: for part (a) of the present exercise, compute average rates of change of $f(x)$ from $x = -1$ to $x = -1 + \Delta x$, for the same three values of Δx as in Exercise 1(a). And so on.)

3. Let $g(x) = -x^3 + 1$.

(a) Show that the average rate of change of $g(x)$ with respect to x, from $x = 4$ to $x = 4 + \Delta x$, is $-48 - 12\Delta x - (\Delta x)^2$. Hint: $-(4 + \Delta x)^3 = -64 - 48\Delta x - 12(\Delta x)^2 - (\Delta x)^3$.

(b) Find $g'(4)$.

4. Let m and b be constants, and let $y = f(x) = mx + b$.

(a) Recalling that the derivative of a function at a point measures the slope of that function at that point, determine, without any computation, what $f'(a)$ should be for any real number a. (Your answer will involve one or more of the constants m and b.) Please explain your answer.

(b) Verify your answer from part (a) using the definition of the derivative. Specifically:

- (i) Compute the average rate of change of $y = f(x)$ from $x = a$ to $x = a + \Delta x$, where a is any real number and Δx is any nonzero number.
- (ii) Use your answer from part (i) to evaluate $f'(a)$.

(c) True or false: for a linear function, average and instantaneous rates of change are always equal. Please explain your answer.

5. Let $D(C)$ be as in Example 2.1.1 above.

(a) Show that the average rate of change of D with respect to C, over the interval $[1,1 + \Delta x]$, is $0.04896 - 0.008\Delta x$.

(b) Use part (a) to find $D'(1)$. Does this result agree (at least to several decimal places) with our conclusion concerning "the instantaneous rate of change of D with respect to C, at $C = 1$," from part (b) of Example 2.1.1 above?

(c) Show that the average rate of change of D with respect to C, over the interval $[2,2 + \Delta x]$, is $0.03296 - 0.008\Delta x$.

(d) Use part (c) to find $D'(2)$. Does this result agree, to several decimal places, with part (b) of Example 2.1.1 (in the case $C = 2$)?

6. Let $g(x) = \cos(x)$.

(a) Show that the average rate of change of $g(x)$, from $x = \pi/2$ to $x = \pi/2 + \Delta x$, is $-\sin(\Delta x)/\Delta x$. Hint: use the trigonometric identity $\cos(\theta + \pi/2) = -\sin(\theta)$.

(b) Use the definition of the derivative to find $g'(\pi/2)$. Hint: use the "trigonometric limit" (2.1.4) above.

7. Let $f(x) = \sqrt{x}$.

(a) Show that the average rate of change of $f(x)$ with respect to x, from $x = 64$ to $x = 64 + \Delta x$, is

$$\frac{\sqrt{64 + \Delta x} - 8}{\Delta x}.$$

(b) Multiply the numerator and denominator of your answer from part (a) by $\sqrt{64 + \Delta x} + 8$, and then do some algebra to simplify, to show that the average rate of change from part (a) equals

$$\frac{1}{\sqrt{64 + \Delta x} + 8}.$$

(c) Use your result from part (b) to show that $f'(64) = 1/16$.