Activity: Prediction using SIR models **Math 1310: CLS**

Goal: To use the SIR model to predict (approximately) the evolution of a disease

Recall: we saw in class that the spread of a disease can, under appropriate assumptions, be modeled by the SIR equations

$$\begin{aligned} S' &= -aSI, \\ I' &= \quad aSI - bI, \\ R' &= \qquad\qquad bI. \end{aligned}$$

Here S, I, and R denote the number of individuals susceptible, infected, and recovered, respectively, at any given time t. We agree that t is measured in days, and that S, I, R are measured in people. Then S', I', and R', which denote the rates of change of S, I, and R respectively, are measured in people per day. Finally, a and b are positive parameters (constants), called the *transmission coefficient* and the *recovery coefficient*, respectively. The units of b are day^{-1}; the units of a are (person-day)$^{-1}$.

1. Fill in the blanks (there are four of them). To make things concrete, we will choose particular values for a and b, let's say $a = 0.00002$ and $b = 0.1$. So the above SIR equations now look like this:

$$\begin{aligned} S' &= -0.00002SI, \\ I' &= \underline{\qquad\qquad} SI - 0.1I, \\ R' &= \qquad\qquad\quad \underline{\qquad\quad} I. \end{aligned}$$

If we're going to predict how the disease evolves, we'd better know how it starts. So let's assume it starts with 95,000 susceptible, 5,000 infected, and none recovered. That is, we assume:

$$S(0) = 95000, \quad I(0) = \underline{\qquad\qquad\qquad}, \quad R(0) = \underline{\qquad\qquad\qquad}.$$

2. How large is the overall population we're working with? Please explain.

We are now going to try and construct a *table* of (approximate) values of S, I, R, S', I', and R', over the course of a few days. Here's the start of that table.

Estimates over the first four days

t	S	I	R		S'	I'	R'
0	95,000	5,000	0				
2							
4							

3. In the above table, fill in the values of S', I', and R' at $t = 0$. (Use the SIR equations in exercise 1 of this tutorial.) Use the space below for scratch if you need to.

4. Fill in the blanks (there are three of them). Suppose we want to know (or approximate) $S(2)$. Since we *know* $S(0)$, all we have to do, to figure out $S(2)$, is to figure out the total *change* in S from $t =$____________ to $t =$____________. Let's call this change ΔS (pronounced "delta S"). Then we know (fill in the blank with a NUMBER, taken from your table above) that

$$S(2) = S(0) + \Delta S = \_\_\_\_\_\_\_\_\_\_\_\_ + \Delta S.$$

5 Fill in the blanks (there are four of them). Let's suppose that the *rate of change* S' of S doesn't vary too much over the first two days. Then we can assume that this rate of change over the first two days is equal to the value of S' at the *start* of these two days. That is, we can assume that, over the first two days, S' is roughly equal to (fill in the blank with a NUMBER, taken from your table above)

$$S'(0) = \_\_\_\_\_\_\_\_\_\_\_\_ \text{ people per day.}$$

NOW: note that the total change ΔS, over the first two days, can be computed by taking the *rate of change of* S *per day*, over those two days, and multiplying it by Δt, where Δt denotes the elapsed time (in days). In other words, we have, in the present situation,

$$\Delta S = S'(0) \times \Delta t = \_\_\_\_\_\_\_\_\_\_\_\_ \times \_\_\_\_\_\_\_\_\_\_\_\_ = \_\_\_\_\_\_\_\_\_\_\_\_.$$

(Fill in the blanks with numbers.)

6. Using your answers to the previous two questions, fill in the (approximate) value of S at $t = 2$, in the table on the previous page.

7. Explain why the value you just filled in is (probably) only an *approximate*, and not an *exact*, value.

8. Fill in the blanks (there are five of them). By arguing as in exercises 4–6 above, we can find that, roughly,

$$\begin{aligned} I(2) &= I(0) + \Delta I \\ &= I(0) + (\_\_\_\_\_\_\_\_\_\_\_\_ \times \Delta t) \\ &= \_\_\_\_\_\_\_\_\_\_\_\_ + (\_\_\_\_\_\_\_\_\_\_\_\_ \times \_\_\_\_\_\_\_\_\_\_\_\_) = \_\_\_\_\_\_\_\_\_\_\_\_. \end{aligned}$$

(The last four blanks should be filled in by numbers; the blank before that, by a symbolic quantity.) Put your final answer from here into the correct place in the above table.

9. Find an approximate value of R at $t = 2$. Hint: we're assuming that the total population $S + I + R$ doesn't change. So $S(2) + I(2) + R(2)$ equals $S(0) + I(0) + R(0)$, and you've already computed all of these last six numbers except for $R(2)$.

10. Now that you have approximate values for $S(2)$, $I(2)$, and $R(2)$, you can compute approximate values of $S'(2)$, $I'(2)$, and $R'(2)$, using the SIR equations on the first page. *Do it*, and put your numbers into the table on the first page. (Use the space below for scratch.)

11. Fill in the blanks (there are five of them). Now that you know (approximately) $S(2)$, $I(2)$, $R(2)$, *and* $S'(2)$, $I'(2)$, and $R'(2)$, you can compute (approximately) $S(4)$, $I(4)$, and $R(4)$ at $t = 4$, by repeating the above procedure. For example, you can observe that, roughly, if ΔS now denotes the change in S from $t = 2$ to $t = 4$, then

$$S(4) = S(2) + \Delta S$$
$$= S(2) + (\underline{\hspace{3cm}} \times \Delta t)$$
$$= \underline{\hspace{3cm}} + (\underline{\hspace{3cm}} \times \underline{\hspace{3cm}}) = \underline{\hspace{3cm}}.$$

(The last four blanks should be filled in by numbers; the blank before that, by a symbolic quantity.) Put your final answer here into the correct place in the above table. Repeat this procedure for I and R.

Now suppose we had taken things *one day*, instead of *two days*, at a time. That is: suppose we had taken $\Delta t = 1$. We would then compute $S(1), I(1), R(1), S'(1), I'(1)$, and $R'(1)$, based on the values of these quantities at $t = 0$, and then use the values at $t = 1$ to compute the values at $t = 2$, and so on.

12. How would the values at $t = 2$ and $t = 4$, computed using $\Delta t = 1$, compare to those you computed previously, using $\Delta t = 2$? Which method is likely to give a better approximation to the *true* nature of the disease after two, or four, days? Please explain. (You don't have to actually compute anything.) Use the space on the back of this page if you need to.