Taylor polynomials.

Recall: if $f(x)$ is differentiable at $x=a$, then the linear approximation to $f(x)$ at $x=a$, given by

$$T_1(x) = f(a) + f'(a)(x-a),$$

satisfies:

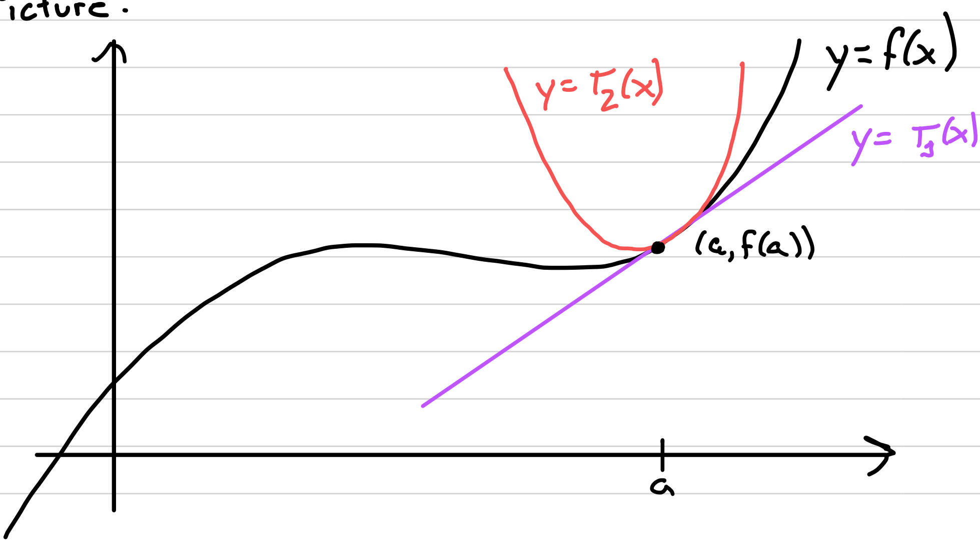
(i) $T_1(a) = f(a) + f'(a)(a-a) = f(a)$;

(ii) $T_1'(x) = \frac{d}{dx}\left[f(a)+f'(a)(x-a)\right] = 0 + f'(a)\cdot 1$

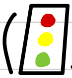
$= f'(a)$, so in particular, $T_1'(a) = f'(a)$.

In sum: T_1 agrees with f, and T_1' agrees with f', at $x=a$.

Picture:



Now suppose we want to approximate $f(x)$, near $x=a$, with a polynomial

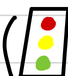
$$T_n(x) = a_0 + a_1(x-a) + a_2(x-a)^2 + a_3(x-a)^3 + a_4(x-a)^4 + \dots + a_n(x-a)^n$$ ()

of degree n. Let's agree: such a polynomial <u>must</u> satisfy

$$T_n^{(k)}(a) = f^{(k)}(a)$$

for all $0 \le k \le n$. That is: The first n derivatives* of T_n and of f agree at $x=a$.

*including the "0^{th} derivatives," meaning just the function values $T_n(a)$ and $f(a)$.

What does this tell us? Well, by (), if $0 \le k \le n$, then the first k derivatives of $T_n(x)$ look like this:

$$T_n'(x) = a_1 + 2a_2(x-a) + 3a_3(x-a)^2 + 4a_4(x-a)^3 + \dots + na_n(x-a)^{n-1}$$
$$T_n''(x) = 2a_2 + 3\cdot 2a_3(x-a) + 4\cdot 3a_4(x-a)^2 + \dots + n(n-1)a_n(x-a)^{n-2}$$
$$T_n^{(3)}(x) = 3\cdot 2a_3 + 4\cdot 3\cdot 2a_4(x-a) + \dots + n(n-1)(n-2)a_n(x-a)^{n-3}$$

$\vdots$ (can you see the pattern?)

$$T_n^{(k)}(x) = k!\,a_k + (\text{something})\cdot(x-a) + \dots + (\text{something})\cdot(x-a)^{n-k}.$$

Now plug in $x=a$, to get this FACT:

$$T^{(k)}(a) = k!\,a_k + (\text{something})\cdot 0 + \dots + (\text{something})\cdot 0 = k!\,a_k.$$

Now remember: we want $T^{(k)}(a)=f^{(k)}(a)$. So by the FACT, we need $k!\,a_k = f^{(k)}(a)$, or

$$a_k = \frac{f^{(k)}(a)}{k!} \qquad (0 \le k \le n).$$

Conclusion:

Suppose the first n derivatives of $f(x)$ exist at $x=a$. Then the n^{th} degree Taylor polynomial

$$T_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$= \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

satisfies

$$T_n^{(k)}(a) = f^{(k)}(a) \quad \text{for } 0 \le k \le n.$$

Example 1. Find the fourth degree Taylor polynomial $T_4(x)$ for $f(x)=\ln(1-x)$ at $x=0$. Also guess what $T_n(x)$ might look like.

Solution. We have $a=0$ and $n=4$. We compute:

$f(0) = \ln(1) = 0.$

$f'(x) = \dfrac{1}{1-x} \cdot -1, \quad f'(0) = -1.$

$$f''(x) = \frac{d}{dx}\left(\frac{-1}{1-x}\right) = \frac{-1}{(1-x)^2}, \quad f''(0) = -1.$$

$$f^{(3)}(x) = \frac{d}{dx}\left(\frac{-1}{(1-x)^2}\right) = \frac{-2}{(1-x)^3}, \quad f^{(3)}(0) = -2.$$

$$f^{(4)}(x) = \frac{d}{dx}\left(\frac{-2}{(1-x)^3}\right) = \frac{-3\cdot 2}{(1-x)^4}, \quad f^{(4)}(0) = -3\cdot 2 = -6.$$

So

$$T_4(x) = 0 - 1\cdot x - \frac{1}{2!}\cdot x^2 - \frac{2}{3!}x^3 - \frac{3\cdot 2}{4!}x^4$$

$$= -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4}.$$

Conjecture: $T_n(x) = -\sum_{k=1}^{n} \frac{x^k}{k}.$