## Mini Project 3: Fermentation, Due Monday, March 18

**For this Mini Project, please make sure you hand in the following, and *only* the following:**

(a) A *cover page,* as described under the "Homework Assignment Guidelines" link from our course page.

(b) The completed *answer page* that you'll find at the end of this handout.

(c) The *graph* of $Y(t)$ requested in exercise 1(b) below.

(d) The *graph* of $Y(t)$ and $A(t)$ (together in a single graph) requested in exercise 2(c) below.

(e) The *graph* of $Y(t)$, $A(t)$, and $S(t)$ (together in a single graph) requested in exercise 3(c) below.

(f) Your complete yeast-alcohol-sugar *Sage code*, as requested in exercise 3(c) below.

**Also, please remember to follow the Mini Project guidelines under the "Homework assignment guidelines" link on our course page.**

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Wine is made by yeast, in the following way. Yeast digests the sugars in grape juice, and produces alcohol as a waste product. This process is called fermentation. The alcohol is toxic to the yeast, though, and the yeast is eventually killed by the alcohol. This stops fermentation, at which point the liquid has become wine, with about 8–12 percent alcohol.

Although alcohol isn't a "species," it acts like a predator on yeast. Unlike certain other predator-prey scenarios, though, the yeast does not have an unlimited food supply. The following exercises develop a sequence of models to take into account the interactions between sugar, yeast, and alcohol.

1. In this exercise, you will build a differential equation for yeast, and plot the solution to this differential equation.

   (a) In this first model, assume that yeast simply grows logistically, with carrying capacity equal to 10 lbs of yeast. Assume that the natural growth rate (also sometimes called the "per capita growth rate") of the yeast is 0.2 lbs of yeast per hour, per pound of yeast. Let $Y(t)$ be the number of pounds of live yeast present after $t$ hours; what differential equation describes the growth of $Y$? Write your answer in the appropriate space on the answer sheet attached at the end of this assignment.

   **Hint: You should reflect on what you know about logistic growth. (See our text, Section 3.5; also see the class notes from Friday, Week 8.) If you're still uncertain, you can consult the program Yeast.sws. The differential equation can be found in the code, if you know where to look.**

   (b) Suppose we start with 0.5 lb of yeast. Graph the solution $Y(t)$ to the differential equation from part (a) above, using the program Yeast.sws, which is available on "The Sage Page" from our course page. Make sure to hand in a printout of this plot.

2. In this next exercise, you will consider the effects of alcohol and yeast on each other.

   (a) Now consider how the yeast produces alcohol. Suppose that waste products of yeast are generated at a rate proportional to the amount of yeast present; specifically, suppose each pound of yeast produces 0.05 lbs of alcohol per hour. (The other major waste product is carbon dioxide gas, which bubbles out of the liquid, and will not be considered further here.) Let $A(t)$ denote the amount of alcohol generated after $t$ hours. Construct a differential equation that describes the growth of $A$. Write your answer in the appropriate space on the answer sheet attached at the end of this assignment.

   (b) Now consider the toxic effect of the alcohol on the yeast. Assume that yeast cells die at a rate proportional to the amount of alcohol present, and also to the amount of yeast present. Specifically, assume that, in each pound of yeast, a pound of alcohol will kill 0.1 lb of yeast per hour. Then, if there are $Y$ lbs of yeast and $A$ lbs of alcohol, how many pounds of yeast will die in one hour? Modify the original logistic equation for $Y$ (from exercise 1 above) to take this effect into account. The modification involves subtracting off a new term that describes the rate at which alcohol kills yeast. What is the new differential equation for $Y'$? Write your answer in the appropriate space on the answer sheet attached at the end of this assignment.

   (c) **To do this part of this exercise, you'll need to modify Yeast.sws.** See the notes directly below this exercise to help you do so.

   You should now have two differential equations, describing the rates of growth of yeast and alcohol. The equations are coupled, in the sense that the yeast equation involves alcohol, and the alcohol equation involves yeast. Assuming that the vat contains, initially, 0.5 lb of yeast and no alcohol, run your Sage code, and print out the resulting graph. On your graph, *make sure you label clearly which curve is $Y$ and which is $A$.* It's OK to label the individual curves by hand, though you're welcome to do it using Sage, if you can figure out how.

**Notes on modifying Yeast.sws.** To do the above part (c) of this exercise, you'll need to:

• Add a line following, and analogous to, the line that reads `Y=0.5`. Your new line should be of the form `A=....` Here, you're specifying the initial amount of alcohol present. See exercise 2(a) above.

• Add a line following, and analogous to, the line that reads `b=10`. Your new line should be of the form `c=....` Here, you're specifying the toxicity coefficient $c$, meaning the rate at which a pound of alcohol kills yeast, per pound of yeast. See exercise 2(b) above.

• Add a line following, and analogous to, the line that reads `Yvalues=[ ]`. Your new line should read `Avalues=[ ]`. You're creating a new list to store the values of A to be computed by your program.

• Add a line following, and analogous to, the line that reads `Yvalues.append(Y)`. Your new line should read `Avalues.append(A)`. You're storing the current value of $A$ into the list called Avalues.

• Rewrite the line that reads `Yprime=k*Y*(1-Y/b)`. Hint: your new differential equation for Y should involve the original term k*Y*(1-Y/b) *and* a new term that involves $Y$, $A$, and the toxicity coefficient, $c$. Read problem 2(b), above, carefully to understand what this new term should look like. (Keep in mind that the effect of this term on Yprime will be *negative*, since it enacts a decrease in Y.)

• Add a line following, and analogous to, the line that begins `Yprime=....` Your new line should be of the form `Aprime=....` Here, you're specifying the differential equation for $A$. To do so, consider the information given in exercise 2(a) above.

• Add a line following, and analogous to, the line that reads `DeltaY=....` Your new line should be of the form `DeltaA=....`

• Add a line following, and analogous to, the line that reads `Y=Y+DeltaY`. Your new line should be of the form `A=....`

• Add a line following, and analogous to, the line that reads

```
Yplot=list_plot(list(zip(tvalues,Yvalues)),plotjoined=True,marker='o',color='blue')
```

Your new line should read

```
Aplot=list_plot(list(zip(tvalues,Avalues)),plotjoined=True,marker='o',color='red')
```

• Replace the last line, which reads

```
show(Yplot,axes_labels=['$t$ (hours)','$Y$ (pounds)'])
```

with the line

```
show(Yplot+Aplot,axes_labels=['$t$ (hours)','$Y,A$ (pounds)'])
```

• Modify all the comment lines (the lines starting with # ) so that they describe what's going on in your NEW program.

**Note:** you *do not* need to print out the code that you end up with upon making the above modifications. But *do* include a copy of the graph generated by that code. (You will be asked to supply some code in exercise 3 below.)

3. Finally, in this problem, you will consider the effect of sugar in the system. To complete part (c) of this exercise, you will need to modify Yeast.sws again. See the notes below this exercise to help you do so.

   (a) The third model will take into account that the sugar in the grape juice is consumed. Suppose the yeast consumes .15 lb of sugar per hour, per lb of yeast. Let $S(t)$ be the amount of sugar in the vat after $t$ hours. Write a differential equation that describes what happens to $S$ over time. Write your answer in the appropriate space on the answer sheet attached at the end of this assignment.

   (b) We will now account for the fact that the carrying capacity for yeast actually *depends* on the amount of sugar present (so that this carrying capacity now varies with time).

   Specifically, let's assume that the carrying capacity for yeast, at any given point in time, is $.4S$ lbs, where $S$ is the amount of sugar present at that time. Rewrite the logistic equation for $Y$ so that the carrying capacity for $Y$ is $.4S$ lbs, instead of 10 lbs. In this new equation, you should retain the term, developed in exercise 2(b) above, that reflects the toxic impact of alcohol on the yeast. Write your answer in the appropriate space on the answer sheet attached at the end of this assignment.

(c) **To do this part of this exercise, you'll need to further modify the code you ended up with at the end of exercise 2 above.** See the notes directly below this exercise to help you do so.

There are now three differential equations. Using Sage, produce a graph which describes what happens to .5 lbs of yeast that is put into a vat of grape juice which contains 25 lbs of sugar at the start.

**Notes on modifying your code**: again, to do the above part (c) of this exercise, you'll need to make modifications to the code you ended up with at the end of exercise 2 above. These modifications will be quite similar to the modifications you've already made, except: whereas your previous modifications added alcohol $A$ into the mix, your new modifications will regard what happens to sugar $S$. So the code you get, when you're done here, should plot graphs of yeast, alcohol, AND sugar over time. In particular, your last line of code should read

```
show(Yplot+Aplot+Splot,axes_labels=['$t$ (hours)','$Y,A,S$ (pounds)'])
```

**Remark:** Note that, in your code, your carrying capacity will depend on $S$. Perhaps the best way to code this is:

• Change the value of b that you used in exercise 2 above to $b = 0.4$;

• In the differential equation that you had for Yprime in the previous two problems, replace the quantity Y/b by Y/(b*S). [Make sure to include the parentheses in the denominator, or Sage will think you mean (Y/b)*S.]

Also, to get full credit, your new program should have modified comment lines (lines starting with #) to describe the new features. Don't forget to print out and include a copy of your graph, with the quantities $Y$, $A$, and $S$ clearly labeled. Also, please include a copy of the completed code you used to generate this graph.

**Mini Project 3 answer sheet**

Please answer the questions using notation we've been using in class (NOT Sage notation), and specify units on all parameters. For example, it's OK to write something like

$$S' = b\,S\,Y, \text{ where } b = 0.0004 \text{ (lb of yeast} \times \text{hour)}^{-1},$$

but *not* just

$$S' = 0.0004\,S\,Y$$

(this doesn't specify the units), or

$$\texttt{Sprime} = 0.0004 * \texttt{S} * \texttt{Y}$$

(this is "Sage-ey" notation, not math notation. Also it doesn't specify units).

Remember: you can always figure out units on parameters by making sure that units on both sides of your differential equations match up. Also: a lb of yeast is not the same as a lb of sugar or alcohol, so please be sure that the distinction is clear in your units.

1(a) $Y' =$ ________________________________________

2(a) $A' =$ ________________________________________

2(b) $Y' =$ ________________________________________

3(a) $S' =$ ________________________________________

3(b) $Y' =$ ________________________________________