CLS Solutions to Selected Exercises: Homework Assignment #1

Assignment. Part 3: A Measles Epidemic (pages 25–26): Exercises 14–20. **Part 4: Other Diseases** (page 26): Exercises 21ab, 22. **Part 6: The basic reproduction number and herd immunity** (pages 29–30): Exercises 30, 32.

Exercise 14.

One stays infected for 14 days. This is because, as discussed in Section 1.2, the recovery coefficient b equals the reciprocal of the number of days k that one stays infected. Here we have $b = 1/14$ day^{-1}, so $k = 14$ days.

Exercise 15.

$S_T = b/a = (1/14)/0.00001 = 7{,}142.9$ individuals.

Exercise 16.

$S(1) \approx 44{,}446.6$, $I(1) \approx 2{,}903.4$, $R(1) \approx 2{,}650.0$; all units are individuals.

Exercise 17.

$S(2) \approx 43{,}156.1$, $I(2) \approx 3{,}986.5$, $R(2) \approx 2{,}857.4$; all units are individuals.

Exercise 18.

$S(3) \approx 41{,}435.7$, $I(3) \approx 5{,}422.1$, $R(3) \approx 3{,}142.1$; all units are individuals.

Exercise 19.

$S(2) \approx 43{,}493.2$, $I(2) \approx 3{,}706.8$, $R(2) \approx 2800.0$; all units are individuals.

Exercise 20.

(a) The new transmission coefficient is half the old one. So the new coefficient is $a = 0.5 \times 0.00001 = 0.000005$ (individual·day)$^{-1}$.

(b) $S_T = b/a = (1/14)/(0.000005) \approx 14{,}286$ individuals.

Another way to see this is: by cutting the transmission coefficient in half, we double the threshold value S_T of S. (Think about why this makes intuitive sense.)

(c) At the outset of the disease, we have

$$I'(0) = aS(0)I(0) - bI(0) = I(0)(aS(0) - b)$$
$$= (2{,}100)(0.000005 \times 45400 - 1/14) = 326.7$$

individuals/day. That is, I' is initially positive, so I is initially increasing, so the quarantine does *not* eliminate the epidemic.

Exercise 21.

(a) Since $b = 0.08$ day^{-1}, we have that the infection lasts for $k = 1/0.08 = 12.5$ days.

(b) We need I' to be positive; that is, $aSI - bI > 0$. Factoring gives $I(aS - b) > 0$. Since $I > 0$, this means we need $aS - b > 0$, or $aS > b$, or $S > b/a = 0.08/0.00002 = 4{,}000.0$. So there must be more than 4,000 susceptible individuals for the illness to take hold.

Exercise 22.

(a) Since the illness lasts for 4 days, we know that our recovery coefficient, b, should be $b = \frac{1}{4 \text{ days}} = 0.25\frac{1}{\text{days}}$. Since a typical susceptible person meets only about 0.3% of infection population each day, we know that $p = 0.3\% = .003$. Finally, since the infection is transmitted in only one contact out of 6, we know that $q = \frac{1}{6} = 0.167$. Hence our SIR model for this measles-like disease looks like:

$$S' = -aSI = -qpSI = -(0.167)(.003)SI = -0.0005SI$$
$$I' = aSI - bI = qpSI - bI = (0.167)(.003) - 0.25I = 0.0005SI - 0.25I$$
$$R' = bI = 0.25I.$$

(b) We want to know: at what point does I start decreasing – that is, at what point does I' become negative? But

$$I' = 0.0005SI - 0.25I$$
$$= I(0.0005S - 0.25).$$

Since $I \geq 0$ always, I' can only be negative if $0.0005S - 0.25 < 0$, or $0.0005S < 0.25$, or $S < 0.25/0.0005 = 500$. In sum: if $S < 500$, then $I' < 0$, and hence the illness will fade away without becoming an epidemic.

Exercise 32.

(a) $r_0 = kaS(0) = 14 \cdot 0.00001 \cdot 45{,}400 = 6.356$.

(b) We need to immunize a fraction greater than

$$1 - \frac{1}{r_0} = 1 - \frac{1}{6.356} = 0.84266\ldots.$$

That is, we need to immunize more than 84.27% of the initial susceptible population.