First Take-Home Exam, part I: *SIR* using Sage
Due Sunday, September 27, 10:00 PM
Worth 25 points total

Your assignment is to complete the following exercises on pages 27–29, in Section 1.3, of our course text:

Part 5: SIR using Euler's method and Sage (Exercises 23–28)

IMPORTANT GUIDELINES FOR COMPLETING THIS ASSIGNMENT. Please turn in, on your own paper, neatly written, complete answers to **all** questions posed in **all** of the Exercises 23–28. For many of the exercises, you will also be required to turn in some graphs and/or some code. **Instructions on exactly what needs to be handed in, and how**, are in the "Hints and Notes" below. Please read these carefully, and follow them closely.

Hints and Notes:

1. For this assignment you'll need to download, from the email I sent you, the program SIR_2020.sws. You'll then need to upload it to your own Sage account. If you're unsure how to do this, please ask.

2. The easiest way to make copies of your Sage graphs, for printing or saving to a file, is to simply take screenshots.

   However, if you want, you can also save Sage graphs to PDF files.

   For example, in the last cell of your SIR_2020.sws worksheet, there is a command that reads `SIRgraph.save('sirgraph.pdf')`. If you evaluate that cell, it will save the graphical output from the previous cell as a pdf file `sirgraph.pdf`. Clicking on the output link `sirgraph.pdf` will then take you to the pdf output. WARNING: After you click on the link `sirgraph.pdf`, you should reload the web page that you're taken to. If you don't reload, the page may still show a graph from an earlier exercise.

   YOU DO NOT NEED TO TURN IN ANY CODE FOR THIS PART OF THE EXAM UNTIL THE VERY END. See the last note below.

3. **Exercise 23.** Part (a): You *do not* need to turn in any graphs for this part of the exercise.

4. **Exercise 24.** Please *do* turn in a copy of the graph you get when you run the program here (after you've changed the stepsize, as described in this exercise).

5. **Exercise 25.** Please *do* turn in a copy of the graph you get when you run the program here (after you've changed the recovery coefficient b, as described in this exercise).

6. **Exercise 26.** Please *do* turn in a copy of the graph you get when you run the program here (after you've changed the recovery coefficient b *back* to what it was, and also made whatever changes you are asked to make in part (a) of this exercise).

7. **Exercise 27.** Part (a): if you're not sure what changes you need to make to reflect the requested change (to this situation where recovered can become susceptible again), you may want to refer to the in-class Activity from Wednesday, September 2 (under the Week 2 lecture notes from our Canvas page).

   Also, for this exercise, please *do* turn in a copy of the graph you get (after you've modified the code to reflect the requested changes, and have run the new code to generate the new graph).

8. **Exercise 28.** For this exercise, please *do* turn in a copy of the graph you get (after you've modified the code to reflect the requested changes, and have run the new code to generate the new graph).

   Now, once you're done with everything, give your Sage program a name that includes your last name (for example, SIRS_2020_lastname.sws – note the "SIRS" instead of "SIR," to denote the fact that this code illustrates the SIRS model), and share it with me through Sage. To do this, use the "Share" button in Sage, and when it asks you to add collaborators, type in my username, which is just "stade" (without the quotation marks). That's all!

MATH 1310–001: CLS

First Take-Home Exam, Part II

Due Sunday, September 27 at 10:00 PM

Though I may have made use of outside resources for this exam, everything I have written here reflects my own understanding of the material, and is written in my own words.

Name:____

Signature:____

Please read the instructions on the next page **CAREFULLY.**

GOOD LUCK!!

Question:	1	2	3	4	5	6	Total
Points:	12	30	8	9	11	5	75
Score:							

DIRECTIONS

This exam is OPEN EVERYTHING! You can use any notes, electronic/online sources, or human resources you would like. But you must UNDERSTAND what you write in the end, and state it in your own words (and math symbols).

You must **show your work** on every problem of this exam. You will be graded on the quality of your exposition and reasoning! Please write everything out using complete sentences, careful arguments, proper mathematical notation, and so on. Please provide the specified number of decimal places in your answers, where requested.

Please complete all work in the space provided. It is fine to print this exam out and complete it by hand. I recommend using scratch paper until you have a solution you're happy with, and then writing that solution carefully on these pages.

If you have access to a word processing program that is good with math symbols, and you have the technology and expertise to complete this exam that way, that's fine too.

Either way, please turn the exam in through Canvas. (Go to "Assignments" on our Canvas page, Click on "Exam 1, September 25" under "Upcoming assignments," and follow the directions there.) If you have written out your exam by hand, then you can submit either a scan of your exam, or photos of the pages, taken with your phone/tablet/computer. Just MAKE SURE IT'S LEGIBLE or you will lose points.

If you get stuck on a problem, please skip it and then come back. You have plenty of time!!

Please sign below:

I have read and I understand these directions. ______________________________

1. Given the following three functions, find the values and expressions requested below. Please simplify your answers.

$$f(x) = 3 - 5x, \qquad g(x) = \frac{2}{2x+3}, \qquad h(x) = x^2 - x.$$

(a) (3 points) $f(g(0)) =$ $f\left(\frac{2}{2\cdot 0+3}\right) = f(2/3) = 3 - 5\cdot\frac{2}{3}$

$= 3 - \frac{10}{3} = \frac{9}{3} - \frac{10}{3} = \boxed{-1/3}$

Notes:
- leave fractions as fractions.
- use "=" between things that are equal.

(Don't use "$\rightarrow$" in place of "=")

(b) (3 points) $g(h(x)) =$

(c) (3 points) $h(f(x)) =$

(d) (3 points) $h(f(g(-2))) =$

2. Find the indicated derivatives, using formulas and rules from class. You do *not* have to use the definition of the derivative. You do *not* need to explain or simplify your answers.

(a) (3 points) $f'(x)$ if $f(x) = 4x^6 + 6x^4 - 6 \cdot 4^x + 4 \cdot 6^x$.

$f'(x) = 4 \cdot 6x^5 + 6 \cdot 4x^3 - 6\ln(4)4^x + 4\ln(6)6^x$

Note: the derivative of b^x is $\ln(b)b^x$, not xb^{x-1}.

(b) (3 points) $\dfrac{d}{dq}\left[\dfrac{q^2}{17} + 17q^2 + \dfrac{17}{q^2}\right]$

$$= \frac{2q}{17} + 34q - \frac{34}{q^3}$$

Notes:
- don't mistakenly switch from q to x.
- $\dfrac{17}{q^2} = 17q^{-2}$, so $\dfrac{d}{dq}\left[\dfrac{17}{q^2}\right] = \dfrac{d}{dq}\left[17q^{-2}\right]$
$= 17 \cdot (-2q^{-3}) = -\dfrac{34}{q^3}$.

(c) (3 points) $\dfrac{d}{dx}\left[\dfrac{3}{\sqrt{x}} - \dfrac{3}{x^3} - \sqrt[3]{x^2}\right]$

(d) (3 points) $\dfrac{dy}{dx}$ if $y = \sin(\cos(x))$.

(e) (3 points) $g'(x)$ if $g(x) = \sin(\cos(4^{5x^2-3}))$.

We apply the chain rule three times in succession:

$$g'(x) = \frac{d}{dx}\left[\sin(\cos(4^{5x^2-3}))\right]$$

$$= \cos(\cos(4^{5x^2-3})) \cdot \frac{d}{dx}\left[\cos(4^{5x^2-3})\right]$$

$$= \cos(\cos(4^{5x^2-3}))(-\sin(4^{5x^2-3})) \cdot \frac{d}{dx}\left[4^{5x^2-3}\right]$$

$$= -\cos(\cos(4^{5x^2-3}))\sin(4^{5x^2-3}) \cdot \ln(4)\, 4^{5x^2-3} \cdot \frac{d}{dx}\left[5x^2-3\right]$$

$$= -10\ln(4)\cos(\cos(4^{5x^2-3}))\sin(4^{5x^2-3})\,4^{5x^2-3} \cdot$$

(f) (3 points) $\dfrac{d}{dx}\left[3^{4^x}\right]$.

$$\frac{d}{dx}\left[3^{4^x}\right] = \ln(3)3^{4^x} \cdot \frac{d}{dx}[4^x]$$
$$= \ln(3)3^{4^x} \cdot \ln(4)4^x$$
$$\left(= \ln(3)\ln(4)3^{4^x}4^x\right) \leftarrow \text{optional simplification}$$

(g) (3 points) $\dfrac{d}{dx}\left[2^{3^{4^x}}\right]$.

(h) (3 points) $\dfrac{dz}{dq}$ if $z = 2\cos(\sin(3q)) + 3\sin(\cos(2q))$.

(i) (3 points) $h'(3)$ if $h(x) = f(u)$ where $u = g(x)$, $g(3) = -2$, $g'(3) = -7$, and $f'(-2) = 5$.

$h(x) = f(g(x))$

$h'(x) = f'(g(x))g'(x)$

$h'(3) =$ (finish this yourself. Your final answer should be a number).

(j) (3 points) The rate at which the volume of an ice cube is decreasing, at the point where the sidelength is 80 mm (millimeters), if the sidelength of the cube is melting at a rate of 7 mm/min. (Recall that the volume V of a cube, of sidelength s, is given by $V = s^3$.)

Given: $V = s^3$, $s = 80$, $\frac{ds}{dt} = -7$.

Find: $\frac{dV}{dt}$ when $s = 80$.

Solution

$$\frac{dV}{dt} = \frac{dV}{ds} \cdot \frac{ds}{dt} = 3s^2 \cdot (-7)$$

When $s = 80$, $\frac{dV}{dt} = 3(80)^2(-7)$

(finish this.)

3. Nestled in the mountains of Wata, on the banks of the river Takeshi, is a town called Boris. Boris has a population of 100,000 Atsuo. (Atsuo is the plural of Atsuo.) A certain mysterious disease (rumored to be contracted from the rafflesia flower) is spreading through this Atsuo population, according to the usual SIR equations:

$$\begin{aligned} S' &= -aSI, \\ I' &= aSI - bI, \\ R' &= \phantom{-aSI - {}} bI. \end{aligned}$$

Here S, I, and R denote the number of Atsuo that are susceptible, infected, and recovered, respectively, at any given time t. We agree that t is measured in days, and that S, I, and R are measured in individual Atsuo.

It's known that, for this particular disease, it takes an Atsuo 25 days on average to recover. (That is, on average one stays infected for 25 days.)

Also, the following values of S and I are observed (at the beginning of day 20 and then a quarter of a day later):

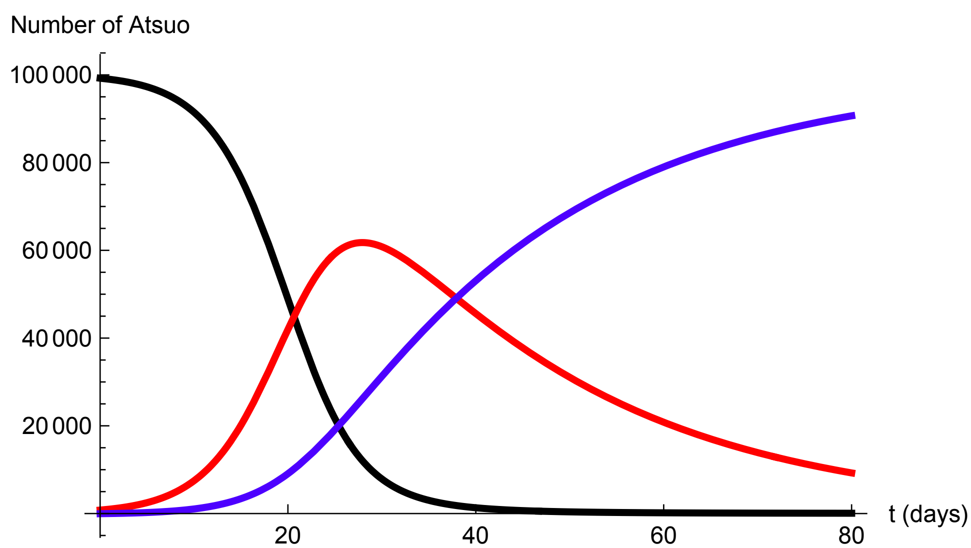
$$\begin{aligned} S(20) &= 48{,}903, \qquad & I(20) &= 42{,}015, \\ S(20.25) &= 47{,}354, \qquad & I(20.25) &= 43{,}162. \end{aligned}$$

Assume throughout this problem that the total number of Atsuo (susceptible + infected + recovered) in Boris always remains the same.

(a) (3 points) Using information given above, find the average rate of change of the susceptible population S from $t = 20$ to $t = 20.25$. Please write your answer to the nearest whole number, and include units in your answer.

(b) (3 points) Find the approximate value of the transmission coefficient a. Hint: use your answer to part (a) of this problem, above, as an approximation to $S'(20)$. Then use one of the SIR equations. Please write your answer to *at least six* decimal places, and include units in your answer.

(c) (2 points) Approximate the threshold value S_T of S **by reading it off of the graph below.** (A rough, eyeball estimate is fine.) (Do NOT use a formula for S_T, though you may use the formula to check your work, if you'd like.) Please show your work and/or explain your answer. Also, please include units in your answer.



$S_T \approx$_______________

4. Let

$$h(x) = x^2 - 3x + 10.$$

(a) (3 points) Find the average rate of change of $h(x)$, from $x = 3$ to $x = 3 + \Delta x$. Simplify your answer as much as possible. (In particular, make sure there's no Δx in the denominator of your answer.)

$$\frac{\Delta y}{\Delta x} = \frac{h(x+\Delta x) - h(x)}{\Delta x}$$

$$= \frac{h(3+\Delta x) - h(3)}{\Delta x}$$

$$\boxed{\text{Note: since this is an average rate of change, there's no "}\lim_{\Delta x \to 0}\text{" anywhere.}}$$

... (show all the algebra needed here)

$$= 3 + \Delta x.$$

(b) (1 point) Fill in the blank (one word): The quantity you found in part (a) equals the slope of the ______________ line to $h(x)$, between $x = 3$ and $x = 3 + \Delta x$.

(c) (2 points) **Use your answer to part (a) above** (do NOT use any differentiation formulas) to find $h'(3)$. Your answer should be a whole number.

$$h'(3) = \lim_{\Delta x \to 0} 3 + \Delta x = 3$$

(d) (1 point) Fill in the blank (one word): The number you found in part (c) equals the slope of the _______________ line to $h(x)$ at $x = 3$.

(e) (2 points) What is the equation of the tangent line to $h(x)$ at $x = 3$?

5. Suppose that the rate of change of sugar dissolving in water is given by the following equation:

$$S' = -\frac{3}{10}S$$

where S is the amount of sugar remaining at a particular time. Assume that sugar is measured in grams and time is measured in minutes.

(a) (1 point) What are the units for S'? Please explain.

$\frac{\text{gram}}{\text{min}}$ because S' is the rate of change of S with respect to t.

(b) (3 points) Assume that $S(0) = 500\text{g}$. Use Euler's method to estimate how much sugar will remain undissolved after 3 minutes, **using a step size of $\Delta t = 3$ minutes.** Include units. Provide an answer to the nearest whole gram.

$$\begin{aligned} S(3) &= S(0) + \Delta S \\ &= S(0) + S'(0)\Delta t \\ &= S(0) + \left(-\frac{3}{10}S(0)\right)\Delta t \end{aligned}$$

(finish this)

(c) (5 points) Estimate how much sugar will remain undissolved after 3 minutes, but this time **with a step size of $\Delta t = 1$ minute.** Include units. Again, provide an answer to the nearest whole gram.

(d) (2 points) Consider your answers from parts (b) and (c) of this problem. Which gives a more accurate estimate of the amount of undissolved sugar in the water after 2 minutes? Explain your answer in terms of stepsize and rates of change.

6. (5 points) Show that the *average* rate of change $\Delta y/\Delta x$ of the function $f(x) = 3^x$, between the points x and $x + \Delta x$, is proportional to $f(x)$ itself. Hint: this is very similar to when we computed the derivative of b^x; see the lecture notes from Tuesday, September 15.