First, Take-Home Exam, part I: *SIR* using Sage

Exercise 23. (b) Beginning: $t = 0$. Ending: $t = 110$. Stepsize: $\Delta t = 0.5$. Total points: $(110 - 0)/.5 + 1 = 221$.

(c) Transmission coefficient: $a = 0.000005$ (individual-day)$^{-1}$. Recovery coefficient: $b = 1/14$ day^{-1}. Initial values: $S(0) =$ 49,600, $I(0) = 400$, $R(0) = 0$ (individuals). On average, one stays infected for 14 days. $S_T = (1/14)/0.000005 =$ 14,285.714... individuals.

(d) First three values of t: 0, 0.5, 1. Last three values: 109, 109.5, 110.

(e) The loop computes successive values of S, I, and R, using the basic ideas that (a) new value = old value + net change and (b) net change = rate of change x stepsize (approximately). The rates of change here are computed using the SIR equations. The lines of code in the loop are being executed 220 times. I estimate that I could maybe do each iteration of the loop in five minutes, if I were *really* speedy. So the whole thing would take me 5 x 220 = 1105 minutes, or 18 hours, 25 minutes. But probably it would actually take me longer.

(f) We don't have actual formulas for S, I, and R, we only know (approximations to) the values of these functions at isolated points. So we're not really plotting functions, we're really plotting lists of points. For this, we use "list_plot" instead of "plot."

Exercise 24. The new graph looks similar to the previous one, but shifted a bit. We expect Euler's method to give better approximations when we use smaller stepsizes.

Exercise 25. (a) There are several differences. For one thing, I peaks at a higher point in the new graph. Also, note that R does not rise as steeply in the new graph.

(b) $b = 1/28$ means the illness now lasts for 28 days. So the changes in the graph that we saw in part (a) are not surprising: because the duration of infection is now greater (28 instead of 14 days), there's more opportunity for infection to accumulate, hence the larger peak in I. Similarly, a longer duration of infection implies slower recovery of the overall population, and this explains the slower rise of R in the new graph.

(c) The new value of S_T is $S_T = (1/28)/0.000005 = 7142.86...$, which roughly agrees with what we see from the graph.

Exercise 26. (a)(b) Halving the transmission coefficient has a number of effects. For one thing, we see that, in the new graph, the peak in I happens much later, and is not as pronounced. This makes sense: a lower rate of transmission means slower, and less pronounced, infection of the overall population. Similar observations can be made about the changes in S and R.

Exercise 27. (b) We replace the formula for Sprime with: Sprime=-a*S*I+R/10. We replace the formula for Rprime with: Rprime=b*I-R/10. The idea is this: if recovered become susceptible again after 10 days, then on average, we have an extra $R/10$ individuals per day contributing to the rate of increase in S, and an extra $R/10$ individuals per day contributing to the rate of decrease in R.

(c) Instead of S and I levelling off to zero and R levelling off to the entire population, we see that S stabilizes at around 14,000, I stabilizes at around 21,000, and R stabilizes at around 15,000. The idea is this: since you can become susceptible again after recovering, the recovered population keeps feeding back into the susceptible population, which feeds back into the infected population, which feeds back into the recovered population, and so on cyclically.

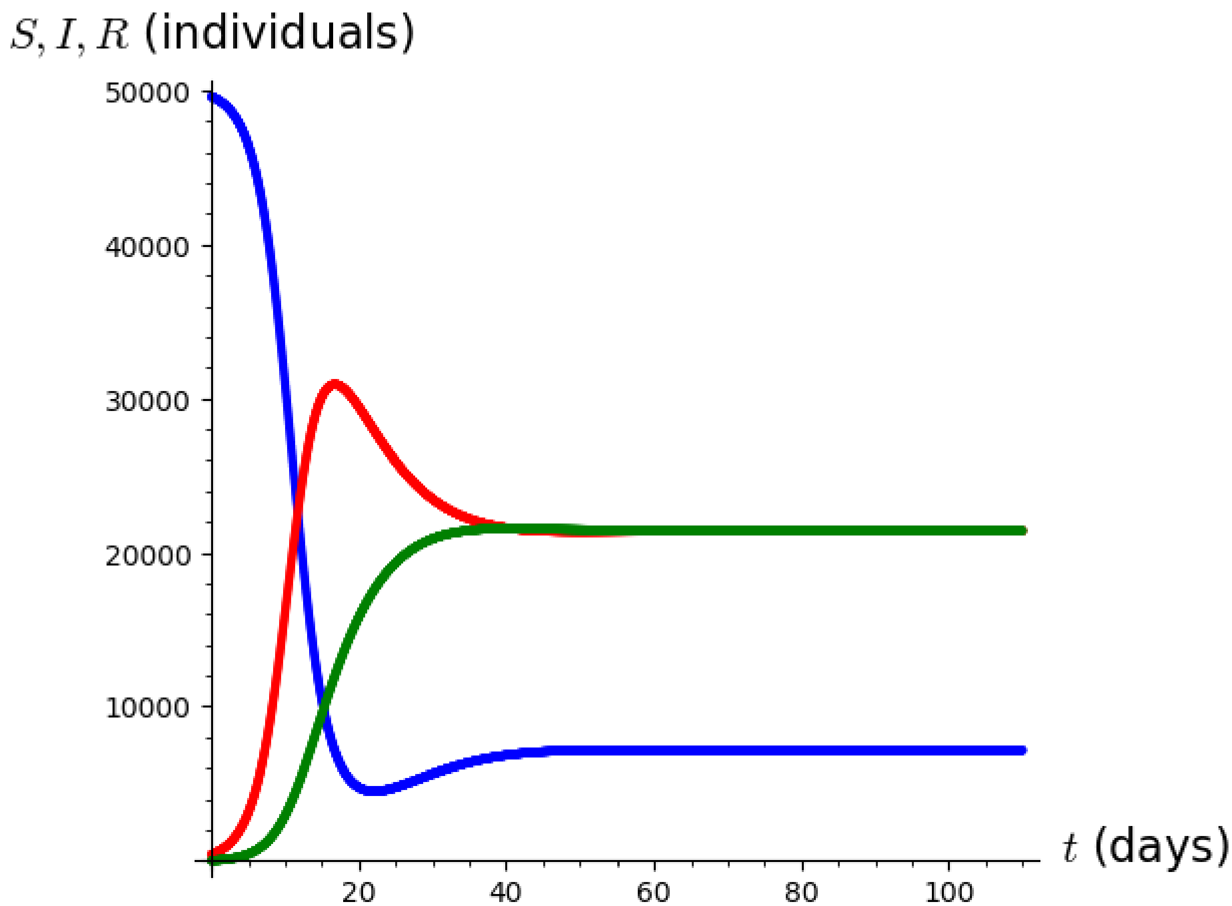
Exercise 28. The period of infection (14 days) is longer than the period of being recovered (10 days), so infection dominates over recovery. To make I and R level off at the same height, you could change b to $1/10$. Then you stay infected for just as long as you stay recovered, so these two sub-populations level off at the same height.

SIRS_2020

SIR_2020 system:sage

```
# SIR program, for studying an epidemic using Euler's method

# First, specify the starting and ending points, stepsize, and total
number of observation points
tstart=0
tfin=110
stepsize=0.05
length=((tfin-tstart)/stepsize)+1

# Next, specify values of parameters, and initial values of variables

a=0.00001
b=1/14
S=49600
I=400
R=0
t=tstart

# Set up empty lists for the values we're about to compute

Svalues=[]
Ivalues=[]
Rvalues=[]
tvalues=[]

# The following loop does three things:
# (1) stores the current values of S, I, R, and t into the lists created
above;
# (2) computes the next values of S, I, R using Euler's method;
# (3) increases t by the stepsize

for i in range(length):

    # Store current values

    Svalues.append(S)
    Ivalues.append(I)
    Rvalues.append(R)
    tvalues.append(t)

    # Compute rates of change using SIRS equations

    Sprime=-a*S*I+(1/14)*R
    Iprime=a*S*I-b*I
    Rprime=b*I-(1/14)*R
```

```
    # Net change equals rate of change times stepsize

    DeltaS=Sprime*stepsize
    DeltaI=Iprime*stepsize
    DeltaR=Rprime*stepsize

    # New values equal current values plus net change

    S=S+DeltaS
    I=I+DeltaI
    R=R+DeltaR
    t=t+stepsize

    # Next time through the loop, the above new values play the role of
current values

# Zip the t values with the S/I/R values into lists of ordered pairs,
and create plots of these

Splot=list_plot(list(zip(tvalues,Svalues)),marker='o',color='blue')
Iplot=list_plot(list(zip(tvalues,Ivalues)),marker='o',color='red')
Rplot=list_plot(list(zip(tvalues,Rvalues)),marker='o',color='green')

# Now plot the computed S,I,R values together on a single graph, with
axes labelled appropriately

SIRgraph=Splot+Iplot+Rplot
show(SIRgraph,axes_labels=['$t$ (days)','$S,I,R$ (individuals)'])
```




```
SIRgraph.save('sirgraph.pdf')
```

sirgraph.pdf

MATH 1310–001: CLS
First Take-Home Exam, Part II
Due Sunday, September 27 at 10:00 PM

Though I may have made use of outside resources for this exam, everything I have written here reflects my own understanding of the material, and is written in my own words.

Name:________________SOLUTIONS________________

Signature:________________________________

Please read the instructions on the next page **CAREFULLY.**

GOOD LUCK!!

Question:	1	2	3	4	5	6	Total
Points:	12	30	8	9	11	5	75
Score:							

DIRECTIONS

This exam is OPEN EVERYTHING! You can use any notes, electronic/online sources, or human resources you would like. But you must UNDERSTAND what you write in the end, and state it in your own words (and math symbols).

You must **show your work** on every problem of this exam. You will be graded on the quality of your exposition and reasoning! Please write everything out using complete sentences, careful arguments, proper mathematical notation, and so on. Please provide the specified number of decimal places in your answers, where requested.

Please complete all work in the space provided. It is fine to print this exam out and complete it by hand. I recommend using scratch paper until you have a solution you're happy with, and then writing that solution carefully on these pages.

If you have access to a word processing program that is good with math symbols, and you have the technology and expertise to complete this exam that way, that's fine too.

Either way, please turn the exam in through Canvas. (Go to "Assignments" on our Canvas page, Click on "Exam 1, September 25" under "Upcoming assignments," and follow the directions there.) If you have written out your exam by hand, then you can submit either a scan of your exam, or photos of the pages, taken with your phone/tablet/computer. Just MAKE SURE IT'S LEGIBLE or you will lose points.

If you get stuck on a problem, please skip it and then come back. You have plenty of time!!

Please sign below:

I have read and I understand these directions. ______________________________________

1. Given the following three functions, find the values and expressions requested below. Please simplify your answers.

$$f(x) = 3 - 5x, \qquad g(x) = \frac{2}{2x+3}, \qquad h(x) = x^2 - x.$$

(a) (3 points) $f(g(0)) = f(2/(0+3)) = f(2/3) = 3 - 5 \cdot \frac{2}{3} = 3 - \frac{10}{3} = -\frac{1}{3}$.

(b) (3 points) $g(h(x)) = g(x^2 - x) = \frac{2}{2(x^2 - x) + 3} = \frac{2}{2x^2 - 2x + 3}$.

(c) (3 points) $h(f(x)) = h(3 - 5x) = (3 - 5x)^2 - (3 - 5x) = 9 - 30x + 25x^2 - 3 + 5x = 25x^2 - 25x + 6$.

(d) (3 points) $h(f(g(-2))) = h(f(2/(2 \cdot (-2) + 3))) = h(f(-2)) = h(3 - 5(-2)) = h(13) = 13^2 - 13 = 156$.

2. Find the indicated derivatives, using formulas and rules from class. You do *not* have to use the definition of the derivative. You do *not* need to explain or simplify your answers.

(a) (3 points) $f'(x)$ if $f(x) = 4x^6 + 6x^4 - 6 \cdot 4^x + 4 \cdot 6^x$.

$f'(x) = 24x^5 + 24x^3 - 6\ln(4)4^x + 4\ln(6)6^x$.

(b) (3 points) $\dfrac{d}{dq}\left[\dfrac{q^2}{17} + 17q^2 + \dfrac{17}{q^2}\right] = \dfrac{2q}{17} + 34q - \dfrac{34}{q^3}$.

(c) (3 points) $\dfrac{d}{dx}\left[\dfrac{3}{\sqrt{x}} - \dfrac{3}{x^3} - \sqrt[3]{x^2}\right] = -\dfrac{3}{2\sqrt{x^3}} + \dfrac{9}{x^4} - \dfrac{2}{3\sqrt[3]{x}}$.

(d) (3 points) $\frac{dy}{dx}$ if $y = \sin(\cos(x))$. $\frac{dy}{dx} = -\cos(\cos(x))\sin(x)$.

(e) (3 points) $g'(x)$ if $g(x) = \sin(\cos(4^{5x^2-3}))$.

$g'(x) = -10x\ln(4)4^{5x^2-3}\sin(4^{5x^2-3})\cos(\cos(4^{5x^2-3}))$.

(f) (3 points) $\dfrac{d}{dx}\left[3^{4^x}\right]$.

$\dfrac{d}{dx}\left[3^{4^x}\right] = \ln(3)\ln(4)4^x 3^{4^x}$.

(g) (3 points) $\dfrac{d}{dx}\left[2^{3^{4^x}}\right]$.

$\dfrac{d}{dx}\left[2^{3^{4^x}}\right] = \ln(2)\ln(3)\ln(4)4^x 3^{4^x} 2^{3^{4^x}}$.

(h) (3 points) $\dfrac{dz}{dq}$ if $z = 2\cos(\sin(3q)) + 3\sin(\cos(2q))$.

$\dfrac{dz}{dq} = -6\sin(\sin(3q))\cos(3q) - 6\cos(\cos(2q))\sin(2q)$.

(i) (3 points) $h'(3)$ if $h(x) = f(u)$ where $u = g(x)$, $g(3) = -2$, $g'(3) = -7$, and $f'(-2) = 5$.

$h'(x) = f'(g(x))g'(x)$, so $h'(3) = f'(g(3))g'(3) = f'(-2)(-7) = 5 \cdot (-7) = -35$.

(j) (3 points) The rate at which the volume of an ice cube is decreasing, at the point where the sidelength is 80 mm (millimeters), if the sidelength of the cube is melting at a rate of 7 mm/min. (Recall that the volume V of a cube, of sidelength s, is given by $V = s^3$.)

We have $V = s^3$, so by the chain rule

$$\frac{dV}{dt} = \frac{dV}{ds}\frac{ds}{dt} = 3s^2\frac{ds}{dt}.$$

If $s = 80$ mm and $ds/dt = -7$ mm/min, then we get

$$\frac{dV}{dt} = \frac{dV}{ds}\frac{ds}{dt} = 3(80)^2(-7) = -134{,}400$$

mm^3/min.

3. Nestled in the mountains of Wata, on the banks of the river Takeshi, is a town called Boris. Boris has a population of 100,000 Atsuo. (Atsuo is the plural of Atsuo.) A certain mysterious disease (rumored to be contracted from the rafflesia flower) is spreading through this Atsuo population, according to the usual SIR equations:

$$\begin{aligned} S' &= -aSI, \\ I' &= \quad aSI - bI, \\ R' &= \qquad\qquad bI. \end{aligned}$$

Here S, I, and R denote the number of Atsuo that are susceptible, infected, and recovered, respectively, at any given time t. We agree that t is measured in days, and that S, I, and R are measured in individual Atsuo.

It's known that, for this particular disease, it takes an Atsuo 25 days on average to recover. (That is, on average one stays infected for 25 days.)

Also, the following values of S and I are observed (at the beginning of day 20 and then a quarter of a day later):

$$\begin{aligned} S(20) &= 48{,}903, \qquad & I(20) &= 42{,}015, \\ S(20.25) &= 47{,}354, \qquad & I(20.25) &= 43{,}162. \end{aligned}$$

Assume throughout this problem that the total number of Atsuo (susceptible + infected + recovered) in Boris always remains the same.

(a) (3 points) Using information given above, find the average rate of change of the susceptible population S from $t = 20$ to $t = 20.25$. Please write your answer to the nearest whole number, and include units in your answer.

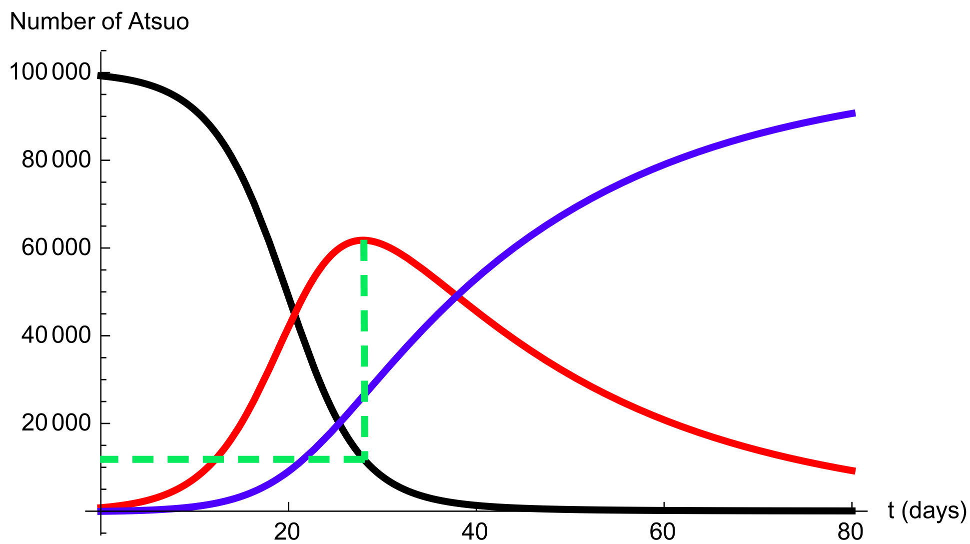
$$\frac{\Delta S}{\Delta t} = \frac{S(20.25) - S(20)}{.25} = \frac{-1549}{0.25} = -6196$$

Atsuo/day.

(b) (3 points) Find the approximate value of the transmission coefficient a. Hint: use your answer to part (a) of this problem, above, as an approximation to $S'(20)$. Then use one of the SIR equations. Please write your answer to *at least six* decimal places, and include units in your answer.

$$a = -\frac{S'(20)}{S(20)I(20)} \approx -\frac{-6196}{48{,}903 \times 42{,}015} = 0.000003 \text{ (Atsuo} \cdot \text{day)}^{-1}.$$

(c) (2 points) Approximate the threshold value S_T of S **by reading it off of the graph below.** (A rough, eyeball estimate is fine.) (Do NOT use a formula for S_T, though you may use the formula to check your work, if you'd like.) Please show your work and/or explain your answer. Also, please include units in your answer.



$S_T \approx$ 12,000 Atsuo.

4. Let

$$h(x) = x^2 - 3x + 10.$$

(a) (3 points) Find the average rate of change of $h(x)$, from $x = 3$ to $x = 3 + \Delta x$. Simplify your answer as much as possible. (In particular, make sure there's no Δx in the denominator of your answer.)

$$\begin{aligned}
\frac{\Delta y}{\Delta x} &= \frac{h(3+\Delta x) - h(3)}{\Delta x} \\
&= \frac{(3+\Delta x)^2 - 3(3+\Delta x) + 10 - (3^2 - 3\cdot 3 + 10)}{\Delta x} \\
&= \frac{9 + 6\Delta x + (\Delta x)^2 - 9 - 3\Delta x + 10 - 10}{\Delta x} \\
&= \frac{3\Delta x + (\Delta x)^2}{\Delta x} = \frac{\Delta x(3+\Delta x)}{\Delta x} \\
&= 3 + \Delta x.
\end{aligned}$$

(b) (1 point) Fill in the blank (one word): The quantity you found in part (a) equals the slope of the ____secant____ line to $h(x)$, between $x = 3$ and $x = 3 + \Delta x$.

(c) (2 points) **Use your answer to part (a) above** (do NOT use any differentiation formulas) to find $h'(3)$. Your answer should be a whole number.

$$h'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} (3 + \Delta x) = 3.$$

(d) (1 point) Fill in the blank (one word): The number you found in part (c) equals the slope of the ____tangent____ line to $h(x)$ at $x = 3$.

(e) (2 points) What is the equation of the tangent line to $h(x)$ at $x = 3$?

$$\begin{aligned} y &= m(x - x_0) + y_0 = h'(3)(x - 3) + h(3) \\ &= 3(x - 3) + (3^2 - 3 \cdot 3 + 10) = 6x - 18 + 19 = 3x + 1. \end{aligned}$$

5. Suppose that the rate of change of sugar dissolving in water is given by the following equation:

$$S' = -\frac{3}{10}S$$

where S is the amount of sugar remaining at a particular time. Assume that sugar is measured in grams and time is measured in minutes.

(a) (1 point) What are the units for S'? Please explain.

The units are grams/minute, since S' is the rate of change of S with respect to time, and S is measured in grams, while time is measured in minutes.

(b) (3 points) Assume that $S(0) = 500$g. Use Euler's method to estimate how much sugar will remain undissolved after 3 minutes, **using a step size of $\boldsymbol{\Delta t} = 3$ minutes.** Include units. Provide an answer to the nearest whole gram.

$$\begin{aligned} S(2) &= S(0) + \Delta S \\ &= S(0) + S'(0)\Delta t \\ &= S(0) + \left(-\frac{3}{10}S(0)\right)\Delta t \\ &= 500 + \left(-\frac{3}{10}\cdot 500\right)\cdot 3 \\ &= 500 - 450 = 50\text{g}. \end{aligned}$$

(c) (5 points) Estimate how much sugar will remain undissolved after 3 minutes, but this time **with a step size of $\Delta t = 1$ minute.** Include units. Again, provide an answer to the nearest whole gram.

$$\begin{aligned} S(1) &= S(0) + \Delta S \\ &= S(0) + S'(0)\Delta t \\ &= S(0) + \left(-\frac{3}{10}S(0)\right)\Delta t \\ &= 500 + \left(-\frac{3}{10}\cdot 500\right)\cdot 1 \\ &= 500 - 150 = 350\text{g}, \end{aligned}$$

so

$$\begin{aligned} S(2) &= S(1) + \Delta S \\ &= S(1) + S'(1)\Delta t \\ &= S(1) + \left(-\frac{3}{10}S(1)\right)\Delta t \\ &= 350 + \left(-\frac{3}{10}\cdot 350\right)\cdot 1 \\ &= 350 - 105 = 245\text{g}, \end{aligned}$$

so

$$\begin{aligned} S(3) &= S(2) + \Delta S \\ &= S(2) + S'(2)\Delta t \\ &= S(2) + \left(-\frac{3}{10}S(2)\right)\Delta t \\ &= 245 + \left(-\frac{3}{10}\cdot 245\right)\cdot 1 \\ &= 245 - 73.5 = 171.5\text{g}. \end{aligned}$$

(d) (2 points) Consider your answers from parts (b) and (c) of this problem. Which gives a more accurate estimate of the amount of undissolved sugar in the water after 3 minutes? Explain your answer in terms of stepsize and rates of change.

The estimate from part (c) is better because the rate of change S' is itself changing, and using a smaller stepsize allows for more frequent recalibration of this rate of change.

6. (5 points) Show that the *average* rate of change $\Delta y/\Delta x$ of the function $f(x) = 3^x$, between the points x and $x + \Delta x$, is proportional to $f(x)$ itself. Hint: this is very similar to when we computed the derivative of b^x; see the lecture notes from Tuesday, September 15.

$$\frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x} = \frac{3^{x+\Delta x} - 3^x}{\Delta x} = \frac{3^x 3^{\Delta x} - 3^x}{\Delta x} = 3^x \left(\frac{3^{\Delta x} - 1}{\Delta x} \right).$$

Note that the quantity $(3^{\Delta x} - 1)/\Delta x$ is constant as far as x is concerned. So we've shown that $\Delta y/\Delta x$ equals a constant times 3^x, and again "equals a constant times" means "is proportional to." So we're done.