1. Recall that the Lotka-Volterra equations, for dual populations of rabbits R and foxes F, are given by:

$$R' = aR\left(1 - \frac{R}{b}\right) - cRF$$

$$F' = dRF - eF$$

(Here, a, b, c, d, and e are positive parameters.) Assume that time is measured in months.

(a) What are the units for the parameter c? Please explain.

(b) Suppose that, initially, the number of rabbits is larger than the carrying capacity of the environment for rabbits. Will the rabbit population initially increase or decrease? Please explain, by referring explicitly to one or more of the above differential equations.

(c) What is the threshold value R_T of R, meaning the value of R at which the fox population peaks (assuming the fox population is initially increasing)? Your answer should involve one or more of the above parameters.

2. (a) Monomers M, dimers D, and trimers T interact in a solution of water. The following kinds of reactions (and only these kinds of reactions) are possible:

____________ Two monomers can spontaneously join to form a dimer.

____________ A monomer and a dimer can spontaneously join to form a trimer.

____________ A trimer can split spontaneously into three monomers.

____________ A dimer can split spontaneously into two monomers.

In the space next to each of the possible reactions above, put the number(s) of the term(s) below that correspond(s) to this reaction. For example, in the first space above, you might conceivably write (vii), (ix) (though this is not the correct answer). (A given reaction can correspond to more than one term in the differential equations.) In filling the four spaces above, you should use *all* of the nine terms below.

All lower case letters below represent positive parameters.

$$\frac{dM}{dt} = \underset{\text{(i)}}{-2aM^2} \underset{\text{(ii)}}{- bMD} \underset{\text{(iii)}}{+ 2eD} \underset{\text{(iv)}}{+ 3gT}$$

$$\frac{dD}{dt} = \underset{\text{(v)}}{aM^2} \underset{\text{(vi)}}{- bMD} \underset{\text{(vii)}}{- eD}$$

$$\frac{dT}{dt} = \underset{\text{(viii)}}{bMD} \underset{\text{(ix)}}{- gT}$$

(b) In the above system of differential equations, what is $M' + 2D' + 3T'$? (Your answer should be a *number.*) Explain what this result means, in terms of the processes at hand.

3. Eating too much cheesecake can make you irritable, because of a complex series of interactions among the cheesecake C, stomach acid A, and *verious grumpimus* V (an enzyme that flows to the brain and makes you cranky).

Suppose C, A, and V interact as follows:

- V grows logistically, with carrying capacity proportional to the amount of stomach acid present;
- In the absence of other factors, stomach acid would grow at a rate proportional to the amount of cheesecake present; however
- This growth of stomach acid is inhibited by *verious grumpimus* – the more there is of this enzyme, the more slowly stomach acid grows;
- Stomach acid dissolves the cheesecake, at a rate proportional to the product of the amount of stomach acid present and the amount of cheesecake remaining;
- Stomach acid is toxic to *verious grumpimus*; V decays at a rate proportional to the square root of the amount of stomach acid present.

Given these assumptions, write down a system of differential equations governing the interactions of C, A, and V. State clearly which quantities in your equations are parameters, and whether each of these parameters is positive or negative.

$C' =$

$A' =$

$V' =$

4. The differential equations we use to model alcohol fermentation are shown below.

$$
\begin{aligned}
Y' &= \underbrace{aY\left(1-\frac{Y}{bS}\right)}_{(i)} \quad - \quad \underbrace{cAY}_{(ii)} \\
A' &= \underbrace{dY}_{(iii)} \\
S' &= \underbrace{-eY}_{(iv)}
\end{aligned}
$$

Match each explanation below to one term of the differential equations above. (Put the correct term number – (i), (ii), (iii), or (iv) – next to each explanation.)

_______ Alcohol is produced at a rate proportional to the amount of yeast.

_______ Yeast dies at a rate proportional to the product of the amount of alcohol and the amount of yeast.

______ Yeast grows logistically, with carrying capacity proportional to the amount of sugar present.

_______ Sugar is consumed at a rate proportional to the amount of yeast.

5. Suppose that a fox population on an island grows according to the logistic differential equation

$$P' = kP\left(1 - \frac{P}{200}\right).$$

(a) If the population is initially 100 foxes, and is initially growing at a rate of 10 foxes per year, find k. Include units in your answer.

(b) True or false: at some point in the distant future (assuming the above logistic model continues to hold), the fox population will be growing more slowly than it is in part (a). Please explain. (You shouldn't need to calculate anything to answer this part.)

(c) Suppose more foxes are brought to the island, so that the total population is now 300 foxes. Will the population increase or decrease after this?
Give a "real world" explanation for your answer.

6. Two types of protein – *sciencus lifus*, denoted S, and *U. arso hilarious*, denoted U – exist in a solution of root beer, denoted R. The differential equations that model the interaction of the two types of protein and the root beer with each other are as follows. (Here, the quantities a, b, d, e, f, g, h denote positive, constant parameters.)

$$S' = \underbrace{aS\left(1 - \frac{S}{bR}\right)}_{(i)}$$

$$U' = \underbrace{\frac{d}{1+eR}}_{(ii)} \quad \underbrace{- fR}_{(iii)}$$

$$R' = \underbrace{gSU}_{(iv)} \quad \underbrace{- hR}_{(v)}$$

Match each explanation below to one term of the differential equations above. (Put the correct term number – (i), (ii), (iii), (iv), or (v) – next to each explanation.) Note: not every term in the above differential equations matches an explanation – that is, there are more terms than explanations.

_______ U consumes root beer at a rate proportional to the amount of root beer present.

______ S grows logistically, with carrying capacity proportional to the amount of R present.

_______ R inhibits the growth of U: the more R there is present, the more slowly U grows.

________ R grows at a rate proportional to the amount of S present times the amount of U present.

7. Populations of Birds B, Geckos G, and Crickets C evolve according to the following differential equations. (Here, B, G, and C are in individual Birds, Geckos, and Crickets respectively; time is measured in days.)

$$B' = \underbrace{aB\left(1 - \frac{B}{dG}\right)}_{\text{(i)}} - \underbrace{eB}_{\text{(ii)}}$$

$$G' = \underbrace{fGC}_{\text{(iii)}} - \underbrace{hB}_{\text{(iv)}}$$

$$C' = \underbrace{\frac{kC}{1 + \ell B}}_{\text{(v)}} - \underbrace{mGB}_{\text{(vi)}}$$

Here a, d, e, f, h, k, ℓ and m are all positive constant parameters.

(a) Match each of the six terms of the differential equations above to one, and only one, of the seven explanations below. (Put the correct term number – (i), (ii), (iii), (iv), (v), or (vi) – next to the explanation that goes with it.) One of the seven explanations does not go with any term, and should be left blank.

_______ Geckos grow at a rate proportional to the product of the numbers of crickets and geckos present.

______ Crickets grows logistically, with carrying capacity proportional to the number of birds present.

_______ Crickets die at a rate proportional to the product of the numbers of geckos and birds present.

______ Birds grows logistically, with carrying capacity proportional to the number of geckos present.

_______ In the absence of other factors, crickets would be born at a rate proportional to the number of crickets present; but the birth rate of crickets is inhibited by growth of the bird population.

_______ The rate at which birds die is proportional to the number of birds present.

_______ Geckos die at a rate proportional to the number of birds present.

(b) Suppose the carrying capacity for birds equals 1/3 the size of the gecko population, and that initially, there are just as many geckos as there are birds. Will the number of birds initially increase or decrease? Please explain.

8. Two proteins, whose concentrations are denoted by P and Q, interact according to the following system of differential equations.

$$\begin{aligned} P' &= \frac{a}{1+bQ} - cP \\ Q' &= \frac{v}{1+wP} - xQ \end{aligned}$$

Here, a, b, c, v, w, and x are positive parameters.

(a) Suppose that, at some point in time, the concentrations P and Q are both increasing. Which of the following must be true at this point in time? Put an "x" in the space by the correct answer.

____________ The growths of both P and Q are speeding up. (That is, their rates of growth are both increasing.)

____________ The growths of both P and Q are slowing down. (That is, their rates of growth are both decreasing.)

____________ The growth of P is speeding up but the growth of Q is slowing down.

____________ The growth of P is slowing down but the growth of Q is speeding up.

(b) The protein P decays at a rate proportional to which of the following variables or parameters? Please circle the correct answer.

$$P \qquad Q' \qquad v \qquad x \qquad Q \qquad \frac{a}{1+bQ}$$

(c) The proteins P and Q are in *equilibrium* with each other when P and Q stop changing. At this point, which *two* of the following quantities must be equal to zero? Please circle both correct answers.

$$P \qquad Q' \qquad v \qquad Q \qquad cP \qquad \frac{a}{1+bQ} \qquad P' \qquad \frac{v}{1+wP}$$

(d) When equilibrium is reached, the following equation must be satisfied:

$$a - cP(1+bQ) = 0.$$

What other, similar equation must be satisfied when equilibrium is reached? Please explain.

(e) If P and Q are measured in milligrams per milliliter (mg/ml), and time is measured in hours (h), then what are the units for the parameter c? Please explain.