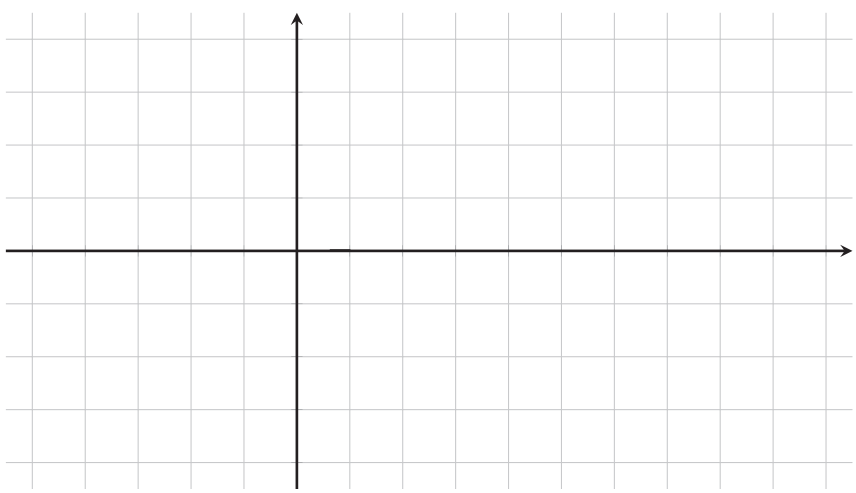
PLEASE NOTE that the first exam will cover all material from Individual Homeworks 1 through 3, and the lecture notes and tutorials from *the first three weeks* of classes. (Sage programming will *not* be on the exam.)

1. Let
$$f(x) = \frac{x^2}{4-x}, \qquad g(x) = 2x, \qquad \text{and} \qquad h(x) = 3 - x.$$
Find the following values, and express in as simple a form as possible. (Expand everything out: for example, $5 \cdot 7 = 35$, $(x^2)^2 = x^4$, $2 + 3(5 + x) = 2 + 15 + 3x = 17 + 3x$, etc.)
   (a) $g(1)$
   (b) $f(g(1))$
   (c) $g(h(x))$
   (d) $h(g(x))$
   (e) $h(h(x))$
   (f) $f(g(h(x)))$

2. Consider the line that passes through $(-4, 3)$ and $(8, -3)$.
   (a) Find the slope of this line.
   (b) Find an equation for this line.
   (c) What is the y-intercept of this line?
   (d) Show (using algebra) that the point $(4, -1)$ lies on this line.
   (e) Graph this line below.



Label the following as part of your graph:
- scale of the graph
- axes
- original two points

- the y-intercept

3. A certain population of size 10,000 is hit with a measles epidemic. The epidemic evolves according to the usual SIR equations

$$\begin{aligned} S' &= -aSI, \\ I' &= \quad aSI - bI, \\ R' &= \qquad\qquad bI. \end{aligned}$$

Here S, I, and R denote the number of individuals susceptible, infected, and recovered, respectively, at any given time t. We agree that t is measured in days, and that S, I, R are measured in people.

On day 50, 100 people are infected and 120 have recovered. (That is, $I(50) = 100$ and $R(50) = 120$.) On day 52, $I = 140$ and $R = 130$. (That is, $I(52) = 140$ and $R(52) = 130$.)

(a) How many susceptibles are there on day 50? Please explain.

(b) Use information from the problem to estimate $R'(50)$. What are the units of $R'(50)$?

(c) Explain the meaning of the number you found in part (b).

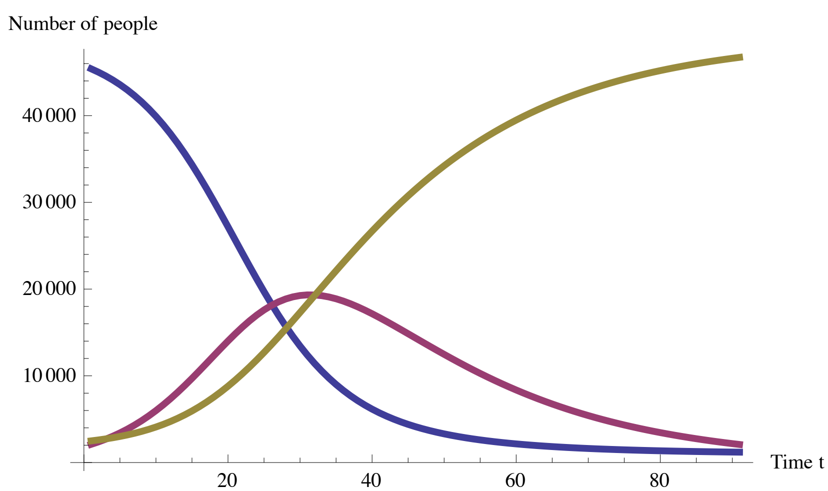
(d) Find the recovery coefficient b. Hint: combine the above equation $R' = bI$ with information above concerning day 50, and solve for b.

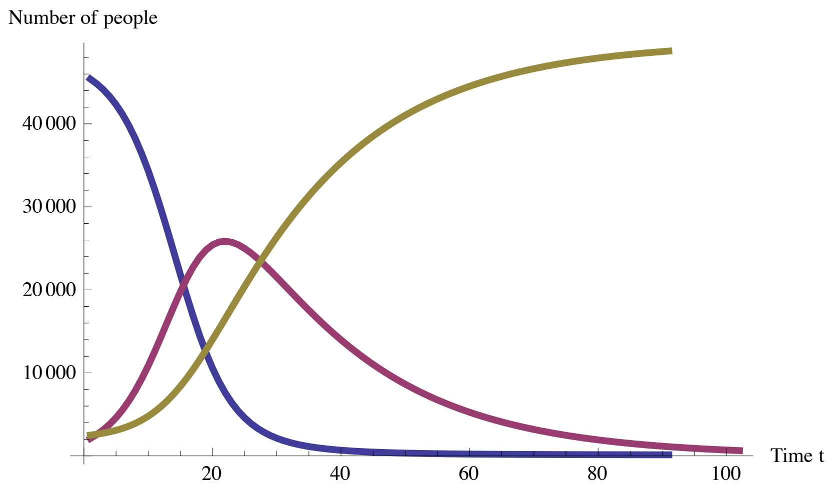
(e) On average, how long does the disease last? Please explain.

(f) Suppose that, at the peak of infection, there are 5,000 susceptibles. What is the transmission coefficient a? Please explain.

4. Below are the graphs of S, I, and R for two different epidemics, both satisfying the usual SIR equations

$$\begin{aligned} S' &= -a\,S\,I, \\ I' &= \quad a\,S\,I - b\,I, \\ R' &= \qquad\qquad b\,I. \end{aligned}$$





(a) In the above two graphs, the recovery coefficients b are the same, but the transmission coefficients a are different. Which of the two graphs – the one on the top or the one on the bottom – corresponds to the *larger* value of a? Please explain.

(b) What (very approximately) is the threshold value S_T of S in each of the above epidemics? Please explain. (You should be able to read this information off of the graphs; you don't need a formula.)

5. Consider a population of bacteria that grows according to the rate equation

$$P' = P - 6,$$

where P is measured in millions of individuals. (We will take hours as our units of time.) We will also assume that $P(0) = 10$ (the population at time $t = 0$ equals 10 million).

Estimate $P(1)$, using steps of size:

(a) one hour;

(b) one half hour.

(c) Which of your two estimates do you think is better? Please explain.

6. Sugar dissolves in water in such a way that the rate of dissolving S' is proportional to the amount S left undissolved.

(a) Write an equation that relates S' and S. Your equation will contain a proportionality constant k. How did you indicate that the sugar is dissolving and not accumulating?

(b) When there are 500g of sugar present, it is dissolving at the rate of 50g/minute. Find k. What are the units of k?

7. Which of the following functions is **not** differentiable (that is, is not locally linear) at $x = 0$? Please circle the correct answer, and explain briefly.

(a) $f(x) = x^3 + 5x + 1$ (b) $h(x) = \begin{cases} -3 & \text{if } x \leq -1, \\ 5 & \text{if } x > -1 \end{cases}$

(c) $g(x) = 10\cos(x)$ (d) $r(x) = |x|$

8. Let $f(x) = x^2 - x$.

   (a) Using your calculator, find the average rate of change $\Delta y/\Delta x$ of $f(x)$ with respect to x, from $x = -1$ to $x = -1 + \Delta x$, for each of the following three values of Δx: $\Delta x = 0.1$, $\Delta x = 0.01$, $\Delta x = 0.001$.

   (b) Using only part (a) of this problem, what do you think $f'(-1)$ is? Please explain.

   (c) Use *algebra* to show that the average rate of change of $f(x)$ with respect to x, from $x = -1$ to $x = -1 + \Delta x$, is $-3 + \Delta x$.

   (d) Find $f'(-1)$.

   (e) Find the equation of the line tangent to the graph of $y = f(x)$ at $x = -1$.

9. Find the indicated derivatives.

   (a) $f'(x)$ if $f(x) = x^{75} + 75x^{10} + x^7 + 7x^6 - 4x^5 - 5x^4 - x - 1$.

   (b) $\dfrac{d}{dx}[\pi]$.

   (c) $g'(w)$ where $g(w) = 4^w + 7 \cdot 3^w$.

   (d) y' if $y = 4\sin(x) - 4\cos(x) - 4^x - x^4 - \pi^4 + 4^\pi$.

   (e) $\dfrac{d}{dx}\left[\sqrt{x} + \dfrac{1}{\sqrt{x}} - \dfrac{1}{x} + \dfrac{17}{x^3} + 12x^{5/2}\right]$.

   (f) $\dfrac{d}{dt}\left[-\dfrac{a}{2}t^2 + bt + c\right]$, where a, b, and c are constants.

   (g) $\dfrac{d}{dz}\left[\pi^2 \tan(z)\right]$.