MATH 2300–888R: Calculus II
Second Midterm Exam
Due Monday, April 6 at 1:00 PM

Though I may have made use of outside resources for this exam, everything I have written here reflects my own understanding of the material, and is written in my own words.

Name: SOLUTIONS

Signature: ____________________

Please read the instructions on the next page CAREFULLY.

Breathe. GOOD LUCK!!

Question:	1	2	3	4	5	6	7	8	9	10	Total
Points:	10	10	10	10	10	10	10	10	10	10	100
Score:											

DIRECTIONS

This exam is OPEN EVERYTHING! You can use any notes, electronic/online sources, or human resources you would like. But you must UNDERSTAND what you write in the end, and state it in your own words (and math symbols).

You will be graded on the quality of your exposition and reasoning! Please write everything out using complete sentences, careful arguments, proper mathematical notation, and so on.

Please complete all work in the space provided. It is fine to print this exam out and complete it by hand. I recommend using scratch paper until you have a solution you're happy with, and then writing that solution carefully on these pages.

If you have access to a word processing program that is good with math symbols, and you have the technology and expertise to complete this exam that way, that's fine too.

Either way, please turn the exam in through the Dropbox feature of Canvas. (Go to "Assignments" on our Canvas page, Click on "Exam 2, First draft" under "Upcoming assignments," and follow the directions there.) If you have written out your exam by hand, then you can submit either a scan of your exam, or photos of the pages, taken with your phone/tablet/computer. Just MAKE SURE IT'S LEGIBLE or you will lose points.

What you hand in on Monday, April 6 will be your FIRST DRAFT. I will grade these drafts carefully and indicate where you need to make corrections and improvements. I will return your annotated exams to you by the start of class on Friday, April 10. You will need to make the indicated revisions, and get the corrected version back to me, by the start of class on Friday, April 17.

You will receive a grade on both the first draft and the final version.

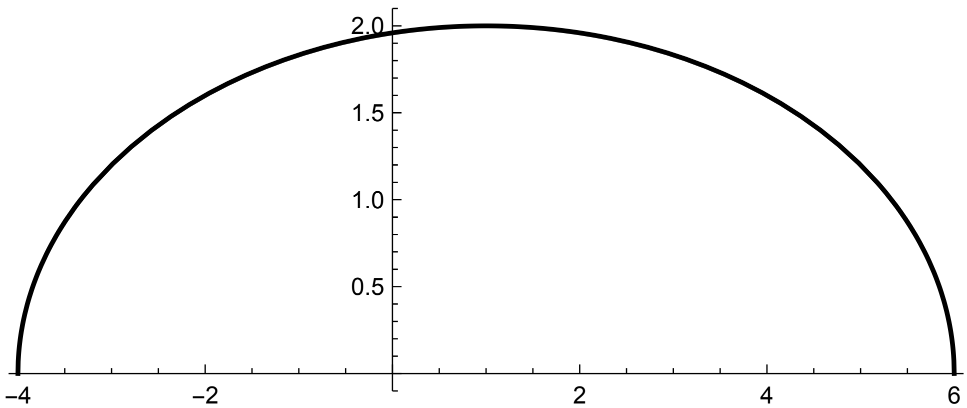
If you get stuck on a problem, please skip it and then come back. You have plenty of time!!

Please sign below:

I have read and I understand these directions. ______________________________

1. A thin lamina (plate), of uniform density δ, occupies the region above the x-axis and below the ellipse

$$4(x-1)^2 + 25y^2 = 100.$$



(a) (5 points) Show that this lamina has mass $m = 5\pi\delta$.

Solution: The mass is given by

$$m = \delta \int_{-4}^{6} \frac{1}{5}\sqrt{100 - 4(x-1)^2}\, dx.$$

Put $x = 1 + 5\sin\theta$, so that $dx = 5\cos\theta\, d\theta$. Also, when $x = -4$, we can choose $\theta = -\pi/2$ (since $1 + 5\sin(-\pi/2) = -4$); when $x = 6$, we can choose $\theta = \pi/2$. So we get

$$\begin{aligned}
m &= \frac{\delta}{5}\int_{-\pi/2}^{\pi/2} \sqrt{100 - 4(5\sin\theta)^2}\,(5\cos\theta\, d\theta) \\
&= \frac{\delta}{5}\int_{-\pi/2}^{\pi/2} 10\sqrt{1-\sin^2\theta}\, 5\cos\theta\, d\theta \\
&= 10\delta \int_{-\pi/2}^{\pi/2} \sqrt{1-\sin^2\theta}\cos\theta\, d\theta = 10\delta\int_{-\pi/2}^{\pi/2} \cos^2\theta\, d\theta \\
&= 5\delta\int_{-\pi/2}^{\pi/2} (1+\cos 2\theta)\, d\theta = 5\delta(\theta + \frac{1}{2}\sin 2\theta)\Big|_{-\pi/2}^{\pi/2} \\
&= 5\delta(\pi/2 + \frac{1}{2}\sin\pi - (-\pi/2 + \frac{1}{2}\sin(-\pi)) = 5\pi\delta.
\end{aligned}$$

(b) (5 points) Find the center of mass $(\overline{x},\overline{y})$ of this lamina.

Solution: It's clear that $\overline{x}$, the x coordinate of the center of mass, is $\overline{x} = 1$, since the lamina is symmetric with respect to the y axis. Also,

$$\begin{aligned} M_x &= \delta \int_{-4}^{6} \frac{(f(x))^2}{2}\, dx = \frac{\delta}{2} \int_{-4}^{6} \left(\frac{1}{5} \sqrt{100 - 4(x-1)^2} \right)^2 dx \\ &= \frac{\delta}{50} \int_{-4}^{6} \left(100 - 4(x-1)^2\right) dx = \frac{\delta}{50} \left(100x - 2(x-1)^3\right) \Big|_{-4}^{6} \\ &= \frac{\delta}{50} \left(100 \cdot 6 - \frac{4}{3}(5)^3 - \left(100 \cdot (-4) - \frac{4}{3}(-5)^3\right) \right) = \frac{40}{3}\delta. \end{aligned}$$

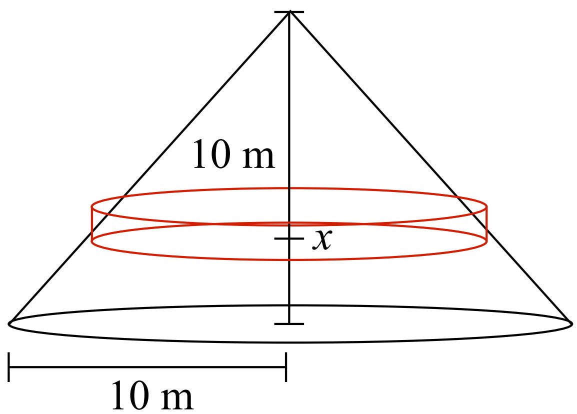
So

$$\overline{y} = \frac{M_x}{m} = \frac{40\delta/3}{5\pi\delta} = \frac{8}{3\pi}.$$

So the center of mass is

$$(\overline{x},\overline{y}) = \left(1, \frac{8}{3\pi}\right).$$

2. (10 points) A tank in the shape of a circular cone, with radius and height equal to 10 m, is full of water.



Find the amount of work required to pump all of the water out of the cone, from a small hole at the very top of the cone. Answer in terms of the density δ of water, in kg/m^3, and the gravitational constant g, in m/s^2. (You don't have to plug in numbers for δ and g.) Please include units in your final answer.

Solution: We need to "add up" (integrate) the work required to pump each infinitesimal slice like the one shown. Let x be the vertical distance from the tip of the cone to this disk. Then this slice has volume $\pi \times \text{radius}^2 \times dx$. The radius is equal to x, by similar triangles, since the radius of the entire cone is the same as its height. So the slice shown has volume $\pi\, x^2\, dx$, and therefore has weight (force) equal to $\pi\,\gamma\,\delta\, x^2\, dx$. The work done in pumping out this slice is therefore $x \times \pi\,\gamma\,\delta\, x^2\, dx = \pi\,\gamma\,\delta\, x^3\, dx$, so the work done in pumping out the entire tank is

$$W = \int_0^{10} \pi\,\gamma\,\delta\, x^3\, dx = \left.\frac{\pi\,\gamma\,\delta\, x^4}{4}\right|_0^{10} = 2{,}500\pi\,\gamma\,\delta.$$

The units are those of weight $\times$ distance, which is to say m^3 $\times$ kg/(m^{2}sec^2) $\times$ m = kg-m^2/sec^2 (or joules, or Newton-meters).

3. In this problem, we are going to use the Integral Test to determine whether

$$\sum_{n=5}^{\infty} \frac{1}{n(\ln(n))^{1/3}}$$

converges or diverges.

(a) (5 points) Fill in the five blanks (you don't need to prove anything): Let

$$f(x) = \frac{1}{x(\ln(x))^{1/3}}.$$

Then $f(x)$ is ___continuous___, ___positive___, and ___decreasing___ for $x \geq 5$, so the conditions of the Integral Test are satisfied. So

the series $\displaystyle\sum_{n=5}^{\infty} \frac{1}{n(\ln(n))^{1/3}}$ and the integral $\displaystyle\int_5^{\infty} \frac{dx}{x(\ln(x))^{1/3}}$

either both ___converge___ or both ___diverge___.

(b) (3 points) Compute the indicated integral:

Solution: Putting $u = \ln(x)$, we find that $du = dx/x$, so

$$\int_5^{\infty} \frac{dx}{x(\ln(x))^{1/3}} = \int_{\ln(5)}^{\infty} \frac{du}{u^{1/3}}$$
$$= \int_{\ln(5)}^{\infty} u^{-1/3}\, du = \frac{3}{2} u^{2/3} \Big|_{\ln(5)}^{\infty} = \infty.$$

(c) (2 points) Fill in the two blanks: the integral ___diverges___, so by the Integral Test, the series ___diverges___ too.

4. In this problem, we are going to use the Integral Test to determine whether

$$\sum_{n=5}^{\infty} \frac{1}{n\ln(n)(\ln(\ln(n)))^3}$$

converges or diverges.

(a) (5 points) Fill in the five blanks (you don't need to prove anything): Let

$$f(x) = \frac{1}{x\ln(x)(\ln(\ln(x)))^3}.$$

Then $f(x)$ is ___continuous___, ___positive___, and ___decreasing___ for $x \geq 5$, so the conditions of the Integral Test are satisfied. So

the series $\displaystyle\sum_{n=5}^{\infty} \frac{1}{n\ln(n)(\ln(\ln(n)))^3}$ and the integral $\displaystyle\int_2^{\infty} \frac{dx}{x\ln(x)(\ln(\ln(x)))^3}$

either both ___converge___ or both ___diverge___.

(b) (3 points) Compute the indicated integral:

Solution: Putting $u = \ln(\ln(x))$, we find that

$$du = \frac{1}{\ln(x)} \cdot \frac{d}{dx}[\ln(x)] = \frac{1}{x\ln(x)}\,dx,$$

so

$$\begin{aligned}\int_5^{\infty} \frac{dx}{x\ln(x)(\ln(\ln(x)))^3} &= \int_{\ln(\ln(5))}^{\infty} \frac{du}{u^3} \\ &= \int_{\ln(\ln(5))}^{\infty} u^{-3}\,du = -\frac{1}{2u^2}\bigg|_{\ln(\ln(5))}^{\infty} = 0 + \frac{1}{2(\ln(\ln(5))^2} < \infty.\end{aligned}$$

(c) (2 points) Fill in the two blanks: the integral ___converges___, so by the Integral Test, the series ___converges___ too.

5. (a) (4 points) Use a test to determine whether

$$\sum_{n=1234}^{\infty} \frac{45n}{n^2+7}$$

converges or diverges. Please state clearly which test you're using, and show clearly how you're using it.

Solution: Let $a_n = 45n/(n^2+7)$ and $b_n = 1/n$. Then $\sum b_n$ diverges, by the p-series test with $p = 1$. Also,

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{45n}{n^2+7} \cdot \frac{n}{1} = \lim_{n\to\infty} \frac{45n^2}{n^2+7} = 45,$$

so by the limit comparison test, $\sum a_n$ diverges too.

(b) (3 points) Use a test to determine whether

$$\sum_{n=4321}^{\infty} \frac{6^n+4^n}{7^n-5^n}$$

converges or diverges. Please state clearly which test you're using, and show clearly how you're using it.

Solution: Let $a_n = (6^n+4^n)/(7^n-5^n)$ and $b_n = (6/7)^n$. Then $\sum b_n$ converges, by the geometric series test with $r = 6/7$. Also,

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{6^n+4^n}{7^n-5^n} \cdot \frac{7^n}{6^n} = \lim_{n\to\infty} \frac{1+(4/6)^n}{1-(5/7)^n} = 1,$$

so by the limit comparison test, $\sum a_n$ converges too.

(c) (3 points) Use a test to determine whether

$$\sum_{n=7}^{\infty} \frac{3n+2}{4n-5}$$

converges or diverges. Please state clearly which test you're using, and show clearly how you're using it.

Solution:

$$\lim_{n\to\infty} \frac{3n+2}{4n-5} = \frac{3}{4} \neq 0,$$

so by the nth term test (also called the divergence test), the series diverges.

6. Consider the series

$$\sum_{n=3}^{\infty} \frac{(-1)^n}{1+\sqrt{n}}.$$

(a) (2 points) Explain carefully how you know that this series converges.

Solution: Write this series as $\sum_{n=3}^{\infty}(-1)^n a_n$, where $a_n = 1/(1+\sqrt{n})$. The a_n's are positive and decreasing, and clearly tend to zero as $n \to \infty$. So the series converges by the alternating series test.

(b) (3 points) Does the above series converge absolutely? Please justify your answer, by stating which test you will use, and applying that test carefully.

Solution: It does not converge absolutely, by limit comparison with $\sum_{n=3}^{\infty} b_n$, where $b_n = 1/\sqrt{n}$. Indeed, $\sum b_n$ diverges by the p-series test with $p = 1/2$. And

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sqrt{n}}{1+\sqrt{n}} = 1,$$

so $\sum a_n$ diverges too.

(c) (3 points) We want to approximate the series on the previous page using the partial sum

$$\sum_{n=3}^{N} \frac{(-1)^n}{1+\sqrt{n}}.$$

How large does N have to be to assure that the error in this approximation is no larger than 0.01? Your answer should be of the form

$$N \geq \text{some integer}$$

(that you write down explicitly).

Solution: By the alternating series remainder estimate, the error involved in approximating an alternating series $\sum_{n=3}^{\infty}(-1)^n a_n$ by a partial sum $\sum_{n=3}^{N}(-1)^n a_n$ is no larger, in magnitude, than a_{N+1}. So we need only choose N so that

$$a_{N+1} = \frac{1}{1+\sqrt{N+1}} \leq 0.01.$$

This is the same as $1+\sqrt{N+1} \geq 100$, or $\sqrt{N+1} \geq 99$, or $N+1 \geq 99^2$, or $N \geq 99^2 - 1 = 9800$.

(d) (2 points) Does the series

$$\sum_{n=3}^{\infty} \frac{(-1)^n}{1+\sqrt{n}}$$

converge to a positive or a negative number, or to zero? Hint: approximate this series with a partial sum that contains *just one summand*. Then use the Alternating Series Remainder Estimate to determine how far off your estimate could possible be.

Solution: It converges to a negative number, for the following reason: if you start with a negative number, and add to that number another number that's smaller in magnitude, then you're never going to get a positive number.
More precisely, here's the argument. We can approximate this series (badly!) with the sum consisting of just the first ($n = 3$) term:

$$\sum_{n=3}^{\infty} \frac{(-1)^n}{1+\sqrt{n}} \approx \sum_{n=3}^{3} \frac{(-1)^n}{1+\sqrt{n}} = \frac{(-1)^3}{1+\sqrt{3}},$$

which is negative. How far off can we possibly be? Well, the alternating series remainder estimate tells us that this crude estimate is off from the actual series value by *less* than the next term, which is $(-1)^4/(1+\sqrt{4})$.
So the actual series value is between

$$\frac{(-1)^3}{1+\sqrt{3}} - \frac{(-1)^4}{1+\sqrt{4}} \quad \text{and} \quad \frac{(-1)^3}{1+\sqrt{3}} + \frac{(-1)^4}{1+\sqrt{4}}.$$

Everything in this range is negative, so the actual series value is negative.

7. (a) (2 points) Give an example of a sequence $\{a_n\}$ such that

$$\lim_{n\to\infty} a_n = 0,$$

$\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ converges, and $\sum_{n=1}^{\infty} a_n$ diverges, or explain why no such example exists.

Solution: Example:

$$a_n = 1/n$$

(the alternating harmonic series converges, but the harmonic series diverges).

(b) (3 points) Give an example of a sequence $\{a_n\}$ such that

$$\lim_{n\to\infty} a_n = 0,$$

$\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ converges, and $\sum_{n=1}^{\infty} a_n$ converges, or explain why no such example exists.

Solution: Example:

$$a_n = 1/n^2$$

(by the p-series test with $p = 2$, $\sum a_n$ converges, and absolute convergence implies convergence, so $\sum(-1)^{n+1}a_n$ converges too).

(c) (3 points) Give an example of a sequence $\{a_n\}$ such that

$$\lim_{n\to\infty} a_n = 0,$$

$\sum_{n=1}^{\infty} a_n$ diverges, and $\sum_{n=1}^{\infty} |a_n|$ converges, or explain why no such example exists.

Solution: This is not possible, because absolute convergence implies convergence.

(d) (2 points) Give an example of a sequence $\{a_n\}$ such that

$$\lim_{n\to\infty} a_n = 1,$$

$\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ converges, and $\sum_{n=1}^{\infty} a_n$ diverges, or explain why no such example exists.

Solution: This is not possible because, by the nth term test, $\lim_{n\to\infty} a_n = 1 \neq 0$ implies that the series diverges.

8. (a) (5 points) Evaluate

$$\sum_{n=6}^{\infty} \frac{1}{(3n-1)(3n+2)}.$$

Solution: It's a telescoping series:

$$\sum_{n=6}^{\infty} \frac{1}{(3n-1)(3n+2)} = \frac{1}{3}\sum_{n=6}^{\infty}\left[\frac{1}{3n-1} - \frac{1}{3n+2}\right]$$
$$= \frac{1}{3}\left(\left[\frac{1}{17} - \frac{1}{20}\right] + \left[\frac{1}{20} - \frac{1}{23}\right] + \left[\frac{1}{23} - \frac{1}{26}\right] + \ldots + \left[\frac{1}{3N-1} - \frac{1}{3N+2}\right] + \ldots\right)$$

Now everything in between the $1/17$ term and the $1/(3N+2)$ term cancels. Since $1/(3N+2) \to 0$ as $N \to \infty$, the series converges to $1/3(1/17) = 1/51$.

(b) (5 points) Evaluate the infinite series

$$\frac{2}{3} - \frac{2}{9} + \frac{2}{27} - \frac{2}{3^4} + \frac{2}{3^5} - \dots.$$

Solution: It equals

$$\begin{aligned}&\frac{2}{3}\left(1 - \frac{1}{3} + \frac{1}{3^2} - \frac{1}{3^3} + \frac{1}{3^4} - \dots\right) \\ &= \frac{2}{3}\sum_{n=0}^{\infty}\left(-\frac{1}{3}\right)^n = \frac{2}{3}\cdot\frac{1}{1-(-1/3)} = \frac{2}{3}\cdot\frac{3}{4} = \frac{1}{2}.\end{aligned}$$

9. In this problem, we will consider convergence of the series

$$\sum_{n=1}^{\infty} \frac{3^n (n!)^2}{(2n)!}.$$

As usual, we will denote the nth term of the series by a_n, that is,

$$a_n = \frac{3^n (n!)^2}{(2n)!}.$$

(a) (4 points) Show that

$$\frac{a_{n+1}}{a_n} = \frac{3(n+1)}{2(2n+1)}.$$

Hint: it may help to observe that

$$(2(n+1))! = (2n+2)(2n+1)(2n)!.$$

Solution:

$$\begin{aligned}
\frac{a_{n+1}}{a_n} &= \frac{3^{n+1}((n+1)!)^2}{(2(n+1))!} \times \frac{(2n)!}{3^n (n!)^2} \\
&= \frac{3^{n+1}}{3^n} \times \frac{((n+1)!)^2}{(n!)^2} \times \frac{(2n)!}{(2(n+1))!} \\
&= \frac{3^{n+1}}{3^n} \times \left(\frac{(n+1)!}{n!}\right)^2 \times \frac{(2n)!}{(2n+2)!} \\
&= 3 \times (n+1)^2 \times \frac{1}{(2n+2)(2n+1)} \\
&= \frac{3 \times (n+1)^2}{(2n+2)(2n+1)} = \frac{3 \times (n+1)^2}{(2(n+1)(2n+1)} \\
&= \frac{3(n+1)}{2(2n+1)}.
\end{aligned}$$

(b) (6 points) Using what was shown in part (a) of this problem, and an appropriate test, determine whether the series $\sum_{n=1}^{\infty} a_n$ on the previous page converges conditionally, converges absolutely, or diverges. State clearly which test you are using, and show clearly how you are using it.

Solution: We apply the ratio test. By part (a) above,

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{3(n+1)}{2(2n+1)}\right| = \frac{3}{4}.$$

This is less than one so, by the ratio test, the series converges.

a

10. (10 points) Does

$$\sum_{n=0}^{\infty} \frac{(3n)!}{25^n (n!)^3}$$

converge or diverge? Please show all your work, and explain completely.

Solution: Similarly to the previous problem, we compute that

$$\begin{aligned}
\frac{a_{n+1}}{a_n} &= \frac{(3(n+1))!}{25^{n+1}((n+1)!)^3} \times \frac{25^n (n!)^3}{(3n)!} \\
&= \frac{25^n}{25^{n+1}} \frac{(3(n+1))!}{(3n)!} \times \frac{(n!)^3}{((n+1)!)^3} \\
&= \frac{1}{25} \frac{(3n+3)!}{(3n)!} \times \left(\frac{n!}{(n+1)!} \right)^3 \\
&= \frac{1}{25} (3n+3)(3n+2)(3n+1) \times \frac{1}{(n+1)^3} \\
&= \frac{(3n+3)(3n+2)(3n+1)}{25(n+1)^3} = \frac{3(n+1)(3n+2)(3n+1)}{25(n+1)^3} \\
&= \frac{3(3n+2)(3n+1)}{25(n+1)^2}.
\end{aligned}$$

But then

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{3(3n+2)(3n+1)}{25(n+1)^2} \right| = \frac{27}{25}.$$

This is larger than one so, by the ratio test, the series diverges.