Maclaurin series.

Suppose $f(x)$ has a power series expansion

$$\left.\begin{aligned} f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots &+ a_{n-1} x^{n-1} \\ &+ a_n x^n \\ &+ a_{n+1} x^{n+1} + a_{n+2} x^{n+2} + \dots \end{aligned}\right] \quad (*)$$

valid on some interval I.

For example (see notes of last time):

(i) $$\frac{1}{3-x} = \frac{1}{3}\left(1 + \frac{x}{3} + \frac{x^2}{3^2} + \frac{x^3}{3^3} + \dots\right) = \sum_{n=0}^{\infty} \frac{x^n}{3^{n+1}}$$

for $-3 < x < 3$,

(ii) $$\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$$

for $-1 < x < 1$.

In general, what can we say about the coefficients $a_0, a_1, a_2, \ldots$??

Well, let's start again with our power series $(*)$:

$$\left.\begin{aligned} f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots &+ a_{n-1} x^{n-1} \\ &+ a_n x^n \\ &+ a_{n+1} x^{n+1} + a_{n+2} x^{n+2} + \ldots \end{aligned}\right] (*)$$

Let's differentiate $(*)$ n times. Since

$$\frac{d}{dx} x^p = p x^{p-1} \quad \text{and} \quad \frac{d}{dx} \text{constant} = 0,$$

we see that:

(a) The n^{th} derivative of
$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots + a_{n-1} x^{n-1}$ is zero,

(b) The n^{th} derivative of $a_n x^n$ is
$n(n-1)(n-2)\cdots 1\, a_n = n! \cdot a_n$;

(c) The n^{th} derivative of
$a_{n+1} x^{n+1} + a_{n+2} x^{n+2} + \ldots$
still has a positive power of x in each term.

Applying (a)(b)(c) to $(*)$ then tells us that:

$f^{(n)}(x) = 0 + n!a_n + $(a sum of terms having positive powers of x)

$$= n!a_n + x \cdot \text{some polynomial.} \qquad (*^{(n)})$$

Now, plug $x=0$ into $(*^{(n)})$, to get:

$$f^{(n)}(0) = n!a_n + 0 \cdot \text{some polynomial},$$

or, solving for a_n:

$$a_n = \frac{f^{(n)}(0)}{n!}.$$

CONCLUSION 1.

If $f(x) = \sum_{n=0}^{\infty} a_n x^n$ on some interval I, then it must be the case that $a_n = f^{(n)}(0)/n!$ for $n = 0, 1, 2, \ldots$.

This fact motivates:

Definition

If $f^{(n)}(0)$ exists for each integer $n \geq 0$, then we call

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

the Maclaurin series for $f(x)$.

Remark. Last time, we saw many examples where $f(x)$ equals its Maclaurin series.

In the following examples, we'll derive Maclaurin series for other functions $f(x)$ – but we don't yet know where, or whether, these functions equal their Maclaurin series.

Examples: find the Maclaurin series for the given function.

Example 1. $f(x) = e^x$.

Solution. The a_n's are given by $a_n = f^{(n)}(0)/n!$ so we need to compute $f^{(n)}(0)$ for all n.

Let's do it:

$$f(x) = e^x,\ f'(x) = e^x,\ f''(x) = e^x, \ldots, f^{(n)}(x) = e^x.$$

So

$$f(0) = e^0 = 1,\ f'(0) = e^0 = 1,\ f''(0) = e^0 = 1, \ldots, f^{(n)}(0) = e^0 = 1.$$

So the Maclaurin series is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Example 2. $f(x) = \sin x$.

Solution. We build a table:

n	0	1	2	3	4	5	6	7
$f^{(n)}(x)$	$\sin x$	$\cos x$	$-\sin x$	$-\cos x$	$\sin x$	$\cos x$	$-\sin x$	$-\cos x$
$f^{(n)}(0)$	0	1	0	-1	0	1	0	-1

So the Maclaurin series is:

$$\frac{f(0)}{0!} + \frac{f'(0)}{1!} + \frac{f''(0)}{2!} + \frac{f^{(3)}(0)}{3!} + \dots$$

$$= \frac{0}{0!} + \frac{1\cdot x}{1!} + \frac{0\cdot x^2}{2!} + \frac{(-1)x^3}{3!} + \frac{0\cdot x^4}{4!} + \frac{1\cdot x^5}{5!} + \frac{0\cdot x^6}{6!} + \frac{(-1)\cdot x^7}{7!} + \dots$$

$$= \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}.$$

Example 3. $\arctan(x)$.

Solution.

The above CONCLUSION 1 tells us: if $f(x) =$ a power series in x, then that series is the Maclaurin series for $f(x)$.

So, by (ii) above, arctan(x) has Maclaurin series

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}.$$