# MORE ON SIR.

I) Recall the SIR equations:

$$\boxed{\begin{aligned} S' &= -aSI \\ I' &= aSI - bI \\ R' &= bI \end{aligned}} \quad \text{(SIR)}$$

$a\ (>0)$: transmission coefficient. Units: $(\text{person-day})^{-1}$.
$b\ (>0)$: recovery coefficient. Units: $\text{day}^{-1}$.

Note: $S$ is decreasing (since $S' < 0$).
$I$ can increase and/or decrease.
$R$ is increasing (since $R' > 0$).

Now, let's use (SIR) for:

II) Prediction. HUGE IDEA:

if $Q$ is any quantity varying with time, then, between any two instants (a "new" one and an "old" one), we have

(a) new $Q$ = old $Q + \Delta Q$,

where $\Delta Q$ is the change in $Q$. Moreover, we have

(b) $\Delta Q = Q' \cdot \Delta t$ *

where $\Delta t$ is elapsed time, and $Q'$ is the rate of change of $Q$.

[(b) says: net change equals rate of change times elapsed time.]

III) Let's now use (a), (b), and (SIR), as follows:

<u>Example.</u> Suppose we know that

$$S(0)=500, \quad I(0)=10, R(0)=0,$$
$$a=0.001, \; b=0.2.$$

(i) Predict $S(2)$, $I(2)$, $R(2)$.

<u>Solution</u>. Let's start with $S$. We have

$$\begin{aligned} S(2) &= S(0)+\Delta S && \text{(by (a) above)} \\ &= S(0)+S'(0)\cdot\Delta t && \text{(by (b) above)} \\ &= S(0)+(-a\cdot S(0)\cdot I(0))\cdot\Delta t && \text{(by (SIR))} \\ &= 500+(-0.001\cdot 500\cdot 10)\cdot 2 && \text{(plug in values)} \\ &= 500-10=490. \end{aligned}$$

We do $R$ next ('cause it's easier than $I$):

$$\begin{aligned} R(2) &= R(0)+\Delta R && \text{(by (a) above)} \\ &= R(0)+R'(0)\cdot\Delta t && \text{(by (b) above)} \\ &= R(0)+b\cdot I(0)\cdot\Delta t && \text{(by (SIR))} \\ &= 0+0.2\cdot 10\cdot 2=4. && \text{(plug in values)} \end{aligned}$$

To find $I(2)$, we recall that $S+I+R$ is constant, and initially equal to $500+10+0=510$, so

$$I(2)=510-S(2)-R(2)=510-490-4=16.$$

Summary: according to the above model,

$$\boxed{S(2)=490, \; I(2)=16, \; R(2)=4} \quad \text{people.}$$

(ii) Use part (i) above to predict $S(4)$, $I(4)$, $R(4)$.

Solution. $S(4) = S(2) + \Delta S$
$= S(2) + S'(2)\Delta t$
$= S(2) + (-a \cdot S(2) \cdot I(2)) \cdot \Delta t$
$= 490 + (-0.001 \cdot 490 \cdot 16) \cdot 2$ } plug in the values computed in Part (i) above
$= 474.32$

Similarly one finds $R(4)$ and $I(4)$; the net result is

$$\boxed{S(4) = 474.32, \quad I(4) = 25.28, \quad R(4) = 10.4.}$$

* NOTE: the "=" in the equation
$\Delta Q = Q'\Delta t$
(see II(b) above) should really be "$\approx$" (approximately equals).
Why? Because Q itself changes with time. More on this idea soon.

## IV) Threshold value of S.

Typically, S will decrease until it can no longer sustain growth in I.
At this point, I peaks.
The value of S at which this happens is called the threshold value $S_T$
(see picture below).

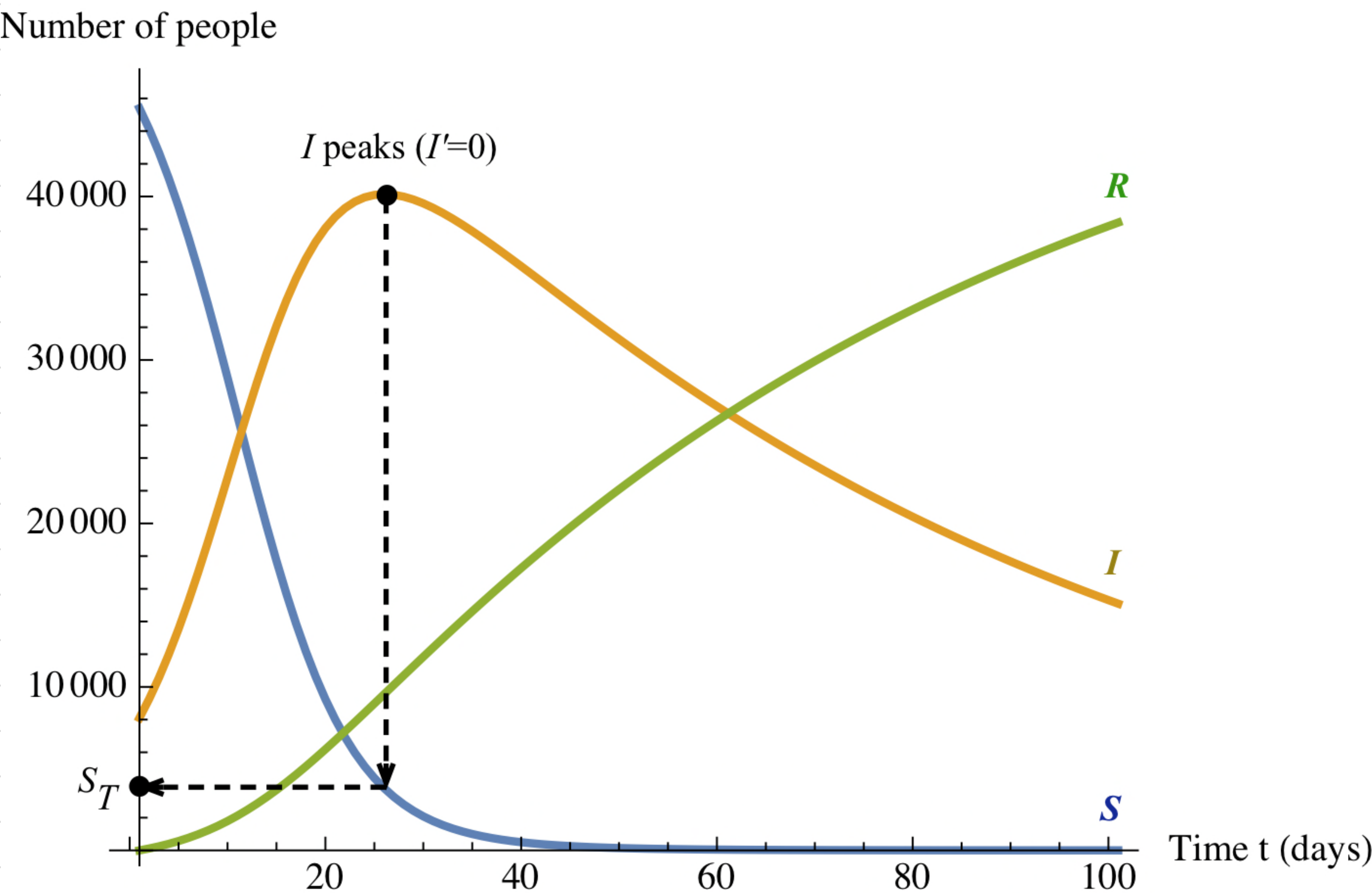
Question: can we compute $S_T$?

Answer: You betcha! We know that, when I peaks, we have $I' = 0$.

Let's emphasize this:

$\boxed{S_T \text{ is the value of } S \text{ where } I'=0.}$ (A)

Picture:



Now by (SIR),

$$I' = aSI - bI,$$

so by (A),

$$aS_T I - bI = 0.$$

Factor:

$$I(aS_T - b) = 0.$$

Divide by $I$:

$$aS_T - b = 0.$$

Solve for $S_T$:

$$\boxed{S_T = b/a.}$$

formula for threshold value $S_T$.

Final note:

the formula

$$\text{new } Q = \text{old } Q + \Delta Q \approx \text{old } Q + Q'\Delta t$$

is really just the linear (= first degree Taylor polynomial) approximation formula:

$$f(a+\Delta x) \approx f(a) + f'(a)\Delta x$$

in disguise!