Taylor series representation of functions.

Today's BIG question:

How, when, where can we say that the Taylor/Maclaurin series
(1) How, (2) when, (3) where

$$T(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

$$= \frac{f(a)}{0!} + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(k)}(a)}{k!}(x-a)^k + \dots$$

equals – that is, converges to – the function $f(x)$??

(1) That is: what tools/techniques do we have?

(2) That is: under what general conditions?

(3) That is: for which values of x?

SOME ANSWERS:

A) Theorem: Taylor Series Remainder Theorem.

Suppose $f^{(k)}(a)$ exists for all $k=0,1,2,\ldots$. Let $R_n(x)$ denote the difference between $f(x)$ and its n^{th} degree Taylor polynomial. That is,

$$f(x) = T_n(x) + R_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + R_n(x). \qquad (*)$$

Then the following are true.

(i) If $\lim_{n\to\infty} R_n(x) = 0$, then $f(x)$ equals its Taylor series:

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

(In particular, the Taylor series converges.)

(ii) (Also called Taylor's inequality.)
Suppose it's known that

$$|f^{(n+1)}(x)| \le M \quad \text{for} \quad |x-a| \le d,^{*}$$

where M and d are positive numbers. Then

$$|R_n(x)| \le \frac{M\,|x-a|^{n+1}}{(n+1)!} \quad \text{for} \quad |x-a| \le d.^{*}$$

* We can replace "for $|x-a| \le d$" with "for all x."

(Proof omitted.)

The essence of the theorem is this:

PART (i):

To show $f(x)$ equals its Taylor series, it suffices to show that the remainder $R_n(x)$ goes to zero as $n \to \infty$.

PART (ii):

If we can bound $f^{(n+1)}(x)$ on the interval where $|x-a| \leq d$, then we get a bound for $R_n(x)$ there. If this bound is good enough, we can deduce that $R_n(x) \to 0$ as $n \to \infty$ on this interval, so by PART (i), $f(x)$ equals its Taylor series there.

Example 1.

(1) Show that the Maclaurin series $T(x)$ for $\sin x$ at $x=0$:

$$T(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1},$$

converges to $\sin x$ for all x.

Solution. First, we need a good bound on $f^{(n+1)}(x)$.

But clearly, $$f^{(n+1)}(x) = \pm \sin x \text{ or } \pm \cos x,$$

depending on n. In any case,

$$|f^{(n+1)}(x)| \leq 1, \text{ for all } x.$$

So by Taylor's inequality,

$$R_n(x) \leq \frac{|x|^{n+1}}{(n+1)!}$$ for all x.

But $\lim_{n \to \infty} \frac{|x|^{n+1}}{(n+1)!} = 0$ * for all x. So by our

*"Factorials grow faster than powers"

Theorem (PART (i)),

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}.$$

NOTE:

PART (ii) of our theorem also has an application to estimation.

Example 2.

Approximate $(3.9)^{-1/2}$, using a third degree Taylor polynomial $T_3(x)$ for

$$f(x) = \frac{1}{\sqrt{5-x}} \qquad \text{at } x=1.$$

At most how far off is your estimate?

Solution.

We saw last time that

$$T_3(x) = \frac{1}{2} + \frac{1}{16}(x-1) + \frac{3}{256}(x-1)^2 + \frac{5}{2048}(x-1)^3.$$

Now $(3.9)^{-1/2} = \dfrac{1}{\sqrt{5-(1.1)}} = f(1.1)$, which is approximately

$T_3(1.1) =$ (we did this last time) $= 0.5063696289 0\ldots$

To estimate the remainder $R_3(1.1)$, we need to estimate $f^{(4)}(x)$, for $|x-1| \le 0.1$.

Now $f^{(4)}(x) = \dfrac{105}{16(5-x)^{9/2}}$ (014).

The largest this can be, for $|x-1| \le 0.1$, is when $5-x$ is smallest, meaning x is largest, meaning $x = 1.1$.

So
$$|f^{(4)}(x)| \le \frac{105}{16(5-(1.1))^{9/2}} \le 0.0143641.$$

So by Taylor's inequality,
$$R_3(1.1) \le \frac{0.0143641}{4!} \cdot |1.1-1|^4 = 6.670 \times 10^{-8}.$$

So our estimate is at most this far off.

(Note: a calculator gives $(3.9)^{-1/2} = 0.5063968354..$

which differs from our estimate $T_3(1.1)$ by

about 5.4×10^{-8}.)