Power series representation of functions.

GOAL: to write various functions as power series.

IT ALL BEGINS with our geometric series formula:

$$\sum_{r=0}^{\infty} r^n = 1 + r + r^2 + \dots = \frac{1}{1-r} \quad \text{for } -1 < r < 1. \qquad (*)$$

From this, we can derive many:

EXAMPLES. Express the following functions as power series.

Example 1: $\dfrac{1}{1-x}$.

Solution. By $(*)$ with x in place of r:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots = \sum_{n=0}^{\infty} x^n \quad \text{for } -1 < x < 1.$$

Example 2: $\dfrac{1}{1+x^2}$.

Solution.
By $(*)$,

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = 1+(-x^2)+(-x^2)^2+(-x^2)^3+\dots$$

$$= 1-x^2+x^4-x^6+\dots = \sum_{n=0}^{\infty}(-1)^n x^{2n}$$

for $-1<-x^2<1$, or

$-1<x<1$.

Example 3. $\dfrac{1}{3-x}$.

Solution. By $(*)$,

$$\frac{1}{3-x} = \frac{1}{3(1-x/3)}$$

$$= \frac{1}{3}\left(1+\frac{x}{3}+\left(\frac{x}{3}\right)^2+\left(\frac{x}{3}\right)^3+\dots\right)$$

$$= \frac{1}{3}\sum_{n=0}^{\infty}\frac{x^n}{3^n}, \qquad \text{for } -1<\frac{x}{3}<1,$$

or $-3<x<3$.

Example 4. $\dfrac{x^4}{3-x}$.

Solution.

By Example 3,

$$\frac{x^4}{3-x} = x^4 \sum_{n=0}^{\infty} \frac{x^n}{3^n} = \sum_{n=0}^{\infty} \frac{x^{4+n}}{3^n},$$

for $-3 < x < 3$.

We can also write this as

$$\frac{x^4}{3-x} = \sum_{n=4}^{\infty} \frac{x^n}{3^{n-4}}, \text{ or } \frac{x^4}{3-x} = 3^4 \sum_{n=4}^{\infty} \frac{x^n}{3^n}, \text{ etc.}$$

FACTS:

- you can also differentiate or integrate power series term-by-term.
- When you do this, if your original power series expression holds for $a < x < b$, then your new one will too.
- All sorts of things, convergence-wise, can happen at the endpoints $x = a$ and $x = b$ when you do this. We won't worry about this much.

Example 5. $\dfrac{7}{(1-x)^2}$.

Solution.

Recall from Example 1 that
$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \qquad \text{for } -1<x<1.$$

Differentiate both sides:

the n=0 term equals zero

$$\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} n x^{n-1} = \sum_{n=1}^{\infty} n x^{n-1} \qquad \text{for } -1<x<1.$$

Multiply by 7:
$$\frac{7}{(1-x)^2} = 7\sum_{n=1}^{\infty} n x^{n-1} \qquad \text{for } -1<x<1.$$

Example 6. $\ln(1-x)$.

Solution.

We know from Example 1 that
$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \qquad \text{for } -1<x<1.$$

Integrate:

$u = 1-x$

$$\int \frac{dx}{1-x} = \int \sum_{n=0}^{\infty} x^n \, dx$$

$$-\ln(1-x) = \sum_{n=0}^{\infty} \int x^n \, dx = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} + C,$$

$$\ln(1-x) = -\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} + C. \qquad (**)$$

To find C, we plug $x=0$ into $(**)$:

$$\ln(1-0) = -\sum_{n=0}^{\infty} \frac{0^{n+1}}{n+1} + C = 0 + C = C, \quad \text{so}$$

$C = \ln(1-0) = 0$. So by $(**)$,

$$\ln(1-x) = -\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots \quad \text{for } -1<x<1.$$

Example 7. $\arctan(x)$.

Solution.

By Example 2,

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n} \qquad \text{for } -1<x<1.$$

So

$$\int \frac{dx}{1+x^2} = \int \sum_{n=0}^{\infty} (-1)^n x^{2n}\, dx$$

$$\arctan(x) = \sum_{n=0}^{\infty} (-1)^n \int x^{2n}\, dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} + C. \quad (*^3)$$

To find C, plug $x=0$ into $(*^3)$:

$$\arctan(0) = \sum_{n=0}^{\infty} \frac{(-1)^n 0^{2n+1}}{2n+1} + C = C,$$

so $C = \arctan(0) = 0$. So, again by $(*^3)$,

$$\arctan(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} \qquad \text{for } -1<x<1.$$

but not easy

COOL FACT: this holds for $x=1$ as well.

So

$$\arctan(1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots \qquad (*^4)$$

But $\arctan(1) = \pi/4$. So if we multiply $(*^4)$ by 4, we get

$$\pi = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ldots\right)$$

(a cool expression for π).