4.5 The Fundamental Theorem of Calculus

In Part A of Section 4.2, we saw that accumulation functions are related to differentiation. In Part B of that section, we saw that accumulations functions are related to area under a graph. So, by way of accumulation functions, differentiation is related to area. This result is of sufficient significance that it is called **The Fundamental Theorem of Calculus.** In this section, we state this theorem precisely, and consider it implications and applications.

To start, suppose $E(t)$ is an accumulation function for $p(t)$, on some interval $[a,b]$. Assume for now that $p(t) \geq 0$ on $[a,b]$. We've seen, then, that

(i) $E'(t) = p(t)$ (at points in the interval where $E'(t)$ exists). (See the boxed statement on page 202, and the arguments leading up to it, especially (4.2.2), (4.2.3), and (4.2.4).)

(ii) The area under the graph of $p(t)$, over $[a,b]$, equals the change ΔE in E, over $[a,b]$. (See the boxed statement on page 204, and the arguments leading up to it, especially (4.2.5).)

Statement (ii) can be written a bit more compactly: the area in question is just $\int_a^b p(t)\,dt$, while the change in E, over $[a,b]$, is just the final value of E on that interval, which is $E(b)$, minus the initial value of E on that interval, which is $E(a)$. So (ii) above reads

(ii') $\int_a^b p(t)\,dt = E(b) - E(a)$.

Putting (i) and (ii') together yields a key conclusion:

Under appropriate circumstances, $\boldsymbol{\int_a^b p(t)\,dt = E(b) - E(a)}$**, where** $\boldsymbol{E'(t) = p(t)}$**.**

This key conclusion *is* the Fundamental Theorem of Calculus. Let us restate this theorem with functions f and F instead of p and E, and an independent variable x instead of t, to highlight the fact that the theorem holds in considerable generality, not just in the context of energy and power.

If $\boldsymbol{F'(x) = f(x)}$**, then**

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

The Fundamental Theorem of Calculus

Note that we have not rigorously proved this theorem, but we have given a plausible (we hope) argument as to why it should be true. And it *is* true, in much greater generality than the arguments that led to it. In particular, $f(x)$ does not need to be nonnegative on $[a,b]$. Nor do we even require that $a < b$. And none of the quantities in question need to have anything to do with energy or power.

We do need to take some care: for certain relatively pathological functions f, the theorem may not hold, or even make sense. But we will not concern ourselves with such functions in this book, nor does one generally encounter them in "real life" contexts.

The power of the The Fundamental Theorem of Calculus is that it allows us, in many cases, to evaluate definite integrals without having to use Riemann sums (or geometric arguments).

Specifically: this theorem tells us that, to integrate a function f, we need to find an *antiderivative*, call it F, of f, meaning a function F whose derivative is f. (In short: to integrate, **antidifferentiate!**) Evaluating F at the appropriate places, and performing the indicated subtraction, then gives us the desired integral.

This example illustrates the kind of approach that is required.

Example 4.5.1. Use The Fundamental Theorem of Calculus to evaluate each of the integrals in Example. 4.4.1.

Solution. The integrals in question are

$$\int_0^5 4\,dx, \qquad \int_0^5 4x\,dx, \qquad \int_0^5 4x^2\,dx.$$

To evaluate any one of these integrals using The Fundamental Theorem of Calculus, we need to first find an antiderivative F of the integrand f in question.

For now, the only we have of antidifferentiating is to try and work backwards from known derivative formulas and facts. (Later, we'll consider other techniques, though even these rely, ultimately, on "reversing" differentiation thought processes.) For example: to find an antiderivative of $f(x) = 4$, we ask: what kind of function has a constant derivative? To answer, we recall that the derivative measures slope, and the functions with constant slopes are the linear functions. In particular, we see that $F(x) = 4x$ is an antiderivative of $f(x) = 4$, since $F'(x) = d[4x]/dx = 4 = f(x)$. So, by The Fundamental Theorem of Calculus,

$$\int_0^5 4\,dx = F(5) - F(0) = 4 \times 5 - 4 \times 0 = 20,$$

as we found previously.

To integrate $f(x) = 4x$, we seek a function whose derivative is f. What kind of function "differentiates down" to $4x$? Well, we recall from many differentiation examples that the derivative of x^2 is $2x$. Adjusting by a constant factor, we find that the derivative of $F(x) = 2x^2$ is $f(x) = 4x$. So this is the $F(x)$ we want. Then by The Fundamental Theorem of Calculus,

$$\int_0^5 4x\,dx = F(5) - F(0) = 2 \times 5^2 - 2 \times 0^2 = 50,$$

also agreeing with previous results.

To integrate $f(x) = 4x^2$ we "work backwards" in a similar fashion to find that $F(x) = \frac{4}{3}x^3$ works:

$$F'(x) = \frac{d}{dx}\left[\frac{4}{3}x^3\right] = \frac{4}{3}\frac{d}{dx}[x^3] = \frac{4}{3} \cdot 3x^2 = 4x^2 = f(x).$$

So by The Fundamental Theorem of Calculus,

$$\int_0^5 4x^2\,dx = F(5) - F(0) = \frac{4}{3}\times 5^3 - \frac{4}{3}\times 5^3 = \frac{500}{3},$$

a result that we were *not* able to obtain previously, without resorting to Riemann sums.

Before proceeding further, we introduce some notation that will allow us to write our answers more immediately.

Definition 4.5.1. If F is a function defined at both $x = a$ and $x = b$, then we write

$$F(x)\Big|_a^b$$

to denote the difference $F(b) - F(a)$.

The Fundamental Theorem of Calculus now tells us that, if F is an antiderivative of f, then

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b.$$

This new notation allows us to write out solutions to integrals compactly. For example, we can now write

$$\int_0^5 4x^2\,dx = \frac{4}{3}x^3\Big|_0^5 = \frac{4}{3}\cdot 5^3 - \frac{4}{3}\cdot 0^3 = \frac{500}{3},$$

without ever having to write something like "let $F(x) = 4x^3/3$" explicitly.

Example 4.5.2. Evaluate the following definite integrals, using the Fundamental Theorem of Calculus.

(i) $\int_0^1 e^x\,dx$ (ii) $\int_0^1 e^{7x}\,dx$ (iii) $\int_0^{\pi/4} \cos(2x)\,dx$ (iv) $\int_1^6 \sqrt{x-1}\,dx$ (v) $\int_0^1 e^{-x^2/2}\,dx$

Solution. (i) $\int_0^1 e^x\,dx = e^x\big|_0^1 = e^1 - e^0 = 1.71828$. Here, we've used the fact that e^x is an antiderivative of e^x, since $d[e^x]/dx = e^x$.

(ii) An antiderivative of e^{7x} is $e^{7x}/7$. This may not be immediately obvious; antidiferentiation sometimes requires some "guessing and checking." That is: we might guess, based for instance on part (i) of this example, that an antiderivative of e^{7x} is e^{7x}. But then we'd check that $d[e^{7x}]/dx = 7e^{7x}$. So we have an extra factor of 7; we can compensate for this by dividing our original guess by 7. This works, since $d[e^{7x}/7]/dx = \frac{1}{7}d[e^{7x}]/dx = \frac{1}{7}\cdot 7e^{7x} = e^{7x}$.

So

$$\int_0^1 e^{7x}\,dx = \frac{e^{7x}}{7}\Big|_0^1 = \frac{e^{7\times 1}}{7} - \frac{e^{7\times 0}}{7} = \frac{e^7 - 1}{7} = 156.51902.$$

(iii) $$\int_0^{\pi/4} \cos(2x)\,dx = \frac{\sin(2x)}{2}\Big|_0^{\pi/4} = \frac{\sin(2\cdot\pi/4) - \sin(2\cdot 0/4)}{2} = \frac{\sin(\pi/2) - \sin(0)}{2} = \frac{1-0}{2} = \frac{1}{2}.$$

(iv) Since differentiation of a power decreases the exponent by one, we would expect antidifferentiation to have the reverse effect. So we might guess that an antiderivative of $\sqrt{x-1} = (x-1)^{1/2}$ is $(x-1)^{3/2}$. But

$$\frac{d}{dx}\left[(x-1)^{3/2}\right] = \frac{3}{2}(x-1)^{1/2}\frac{d}{dx}[(x-1)] = \frac{3}{2}(x-1)^{1/2},$$

so our guess is off by a factor of $3/2$. Our new guess of $2(x-1)^{3/2}/3$ then works as an antiderivative of $\sqrt{x-1}$, as you should check. So

$$\int_1^6 \sqrt{x-1}\,dx = \frac{2}{3}(x-1)^{3/2}\bigg|_1^6 = \frac{2}{3}\left((6-1)^{3/2} - (1-1)^{3/2}\right) = \frac{2\cdot 5^{3/2}}{3} = \frac{2\sqrt{125}}{3} = 7.45356.$$

Compare this with the results of Example 4.3.1 (and Exercises 2 and 3, Section 4.3).

(v) To evaluate $\int_0^1 e^{-x^2/2}\,dx$ using The Fundamental Theorem of Calculus, we'd need to find an antiderivative of $e^{-x^2/2}$. But as it turns out, there *is* no such antiderivative that can be written in closed form.

What this example illustrates is that the The Fundamental Theorem of Calculus doesn't always work. It also illustrates how philosophically different antidifferentiation is from differentiation. Specifically, if we can write something down in terms of familiar functions and familiar mathematical operations, then we can differentiate it, in terms of familiar functions and familiar mathematical operations. With antidifferentiation, we're not so lucky, as the function $f(x) = e^{-x^2/2}$ demonstrates.

Fortunately, we always have Riemann sums as a recourse. And in this case we're *very* fortunate, because the integral $\int_0^1 e^{-x^2/2}\,dx$, and similar ones, show up all over the place in probability and statistics. (The function $f(x) = e^{-x^2/2}\,dx$ represents an example of a *normal* curve.) We'd be quite stuck if we weren't able to evaluate such integrals at all.

The Fundamental Theorem of Calculus can be applied to the calculation of accumulation functions, provided that one has a formula for the integrand, and for the antiderivative of this integrand.

Example 4.5.3. Find the amount of work done if a force of $f(x) = 2 + \cos(x)$ pounds is applied from $x = 0$ to $x = 6$ feet.

Solution. The amount of work done equals

$$\int_0^6 (2+\cos(x))\,dx = (2x+\sin(x))\big|_0^6 = (2\times 6+\sin(6)) - (2\times 0+\sin(0)) = 12+\sin(6)$$
$$= 11.7206 \text{ foot-pounds.}$$

Here is another example, which illustrates several ideas.

Example 4.5.4. During the course of a day, from 6 AM to 6 PM, solar cells on a school absorb energy at a rate given by

$$p_1(t) = \frac{75}{1+(t-6)^2}.$$

Here, t denotes the number of hours since 6 AM of the day in question, and p_1 is in kilowatts. During this same time period, the school uses energy at a roughly constant rate, given by $p_2(t) = 15$. The units here are the same as for p_1.

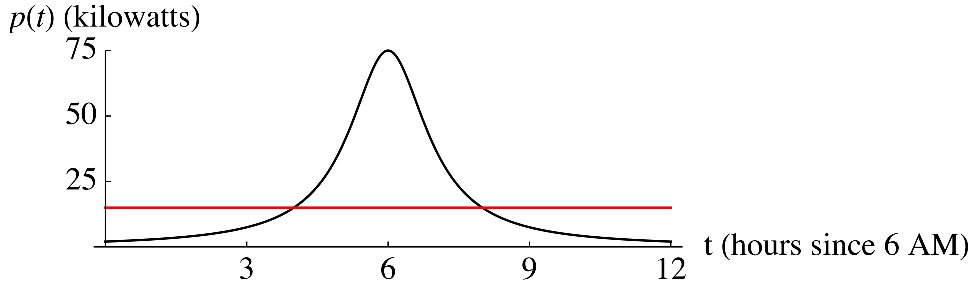

Figure 4.13. Power generated (in black) and consumed (in red)

(i) Set up an integral that measures net energy generated (meaning energy absorbed minus energy used), over the first T hours of the 12-hour period in question.

(ii) Using what you know about the arctangent function, together with some guessing and checking, evaluate the integral.

(iii) Over the entire 12-hour period, does the school experience a net gain, or a net loss, of energy?

Solution. (i) Since energy comes in at a rate of $p_1(t)$, and goes out at a rate of $p_2(t)$, net energy is generated at a rate of $p(t) = p_1(t) - p_2(t)$. (Again, $p_1(t)$, $p_2(t)$, and $p(t)$ are *rates* of energy absorption/usage, so they are measures of *power*.)

Let $E(T)$ denote the net energy generated, from $t = 0$ to $t = T$. We then have

$$E(T) = \int_0^T p(t)\,dt = \int_0^T \left(\frac{75}{1+(t-6)^2} - 15\right) dt. \tag{4.5.1}$$

(ii) The above integral looks a bit daunting, but perhaps less so if we recall that $d[\arctan(t)]/dt = 1/(1+t^2)$. That is, $1/(1+t^2)$ has antiderivative $\arctan(t)$. Using this fact, we might guess that $75/(1+(t-6)^2)$ has antiderivative $75\arctan(t-6)$. And we'd be right:

$$\frac{d}{dt}\big[75\arctan(t-6)\big] = 75\frac{d}{dt}\big[\arctan(t-6)\big] = 75\cdot\frac{1}{1+(t-6)^2}\frac{d}{dt}[t-6] = \frac{75}{1+(t-6)^2}.$$

So (using also the fact that 15 has antiderivative $15t$), equation (4.5.1) gives

$$\begin{aligned}E(T) &= \int_0^T \left(\frac{75}{1+(t-6)^2} - 15\right) dt \\ &= (75\arctan(t-6) - 15t)\Big|_0^T = (75\arctan(T-6) - 15T) - (75\arctan(0-6) - 15\times 0) \\ &= 75\arctan(T-6) - 15T - 75\arctan(-6)\end{aligned}$$

killowatt-hours.

(iii) We have

$$E(12) = 75 \arctan(12-6) - 15 \times 12 - 75 \arctan(-6) = 30.8471 \text{ killowatt-hours.}$$

Since net energy generated is positive, the school generates more energy than it uses, over the 12-hour period.

As Figure 4.13 illustrates, $p_1(t)$ is sometimes larger, and sometimes smaller, than $p_2(t)$. So this example demonstrates, among other things, how accumulation functions and The Fundamental Theorem of Calculus can work even when the function being "accumulated" (in this case, $p(t) = p_1(t) - p_2(t)$) sometimes assumes negative values.

Exercises

Part 1: Formulas for integrals

1. Determine the exact value, to four decimal places, of each of the following integrals.

(a) $\int_0^1 dt$

(Note that $\int_0^1 dt$ is just short for $\int_0^1 1\,dt$.)

(b) $\int_3^7 (2 - 3x + 5x^2)\,dx$

(c) $\int_0^{5\pi} \sin x\,dx$

(d) $\int_0^{5\pi} \sin(2x)\,dx$

(e) $\int_0^1 e^t\,dt$

(f) $\int_1^6 \frac{dx}{x}$

(Note that $\frac{dx}{x}$ is just short for $\frac{1}{x}\,dx$.)

(g) $\int_0^4 (7u - 12u^5)du$

(h) $\int_0^1 2^t\,dt$

(i) $\int_{-1}^1 s^2\,ds$

(j) $\int_{-1}^1 s^3\,ds$

2. Use the Fundamental Theorem of Calculus to compute each of the two integrals of Example 4.4.2. Make sure you get the same answers as were obtained in that example.

3. Find $\int_1^6 \sqrt[3]{x-1}\,dx$, to four decimal places. Hint: following the approach of Example 4.5.2(iv) above, we might guess that an antiderivative of $\sqrt[3]{x-1} = (x-1)^{1/3}$ is $(x-1)^{4/3}$. Check whether this antiderivative really works, by differentiating it. Then adjust your guess if necessary.

4. Express the values of the following integrals in terms of any parameters they contain. Simplify as much as possible. You may need to do some guessing and checking to find the appropriate antiderivatives. **Example:**

$$\int_0^1 e^{bx}\,dx = \left.\frac{e^{bx}}{b}\right|_0^1 = \frac{e^{b\cdot 1}}{b} - \frac{e^{b\cdot 0}}{b} = \frac{e^b - 1}{b}.$$

(a) $\displaystyle\int_3^7 kx\,dx$

(b) $\displaystyle\int_0^\pi \sin(\alpha x)\,dx$

(c) $\displaystyle\int_1^4 (px^2 - x^3)\,dx$

(d) $\displaystyle\int_0^1 e^{ct}\,dt$

(e) $\displaystyle\int_{\ln 2}^{\ln 3} e^{ct}\,dt$

(f) $\displaystyle\int_1^b (5 - x)\,dx$

(g) $\displaystyle\int_0^1 a^t\,dt$

(h) $\displaystyle\int_1^2 u^c\,du$

Part 2: The Fundamental Theorem of Calculus and accumulation functions

5. A space heater runs over a 24-hour period, at a power level of

$$f(t) = 1000\left(1 + \cos\left(\frac{\pi}{12}t\right)\right)$$

watts, where t is the number of hours since midnight (which is $t = 0$).

(a) Write down an integral to measure how much energy this heater uses over the course of the 24-hour period (from midnight to midnight).

(b) Evaluate the integral in part (a), to four decimal places. (Hint: An antiderivative of $\cos(kx)$ is $\sin(kx)/k$, if k is a nonzero constant.) What are the units for your answer?

6. Consider the car of Example 4.2.1. Suppose the speed function $s(t)$ depicted there has formula

$$s(t) = 127t - 90t^2 + 17.35t^3 + 5t^4 - 2.079t^5 + 0.18t^6.$$

Use the Fundamental Theorem of Calculus to find the distance traveled by the car, over the 5-hour period. Check your answer against the estimate we obtained in that example, to make sure you're in the right ballpark.

7. There's a difference between speed and velocity: the first measures how fast you're going, while the second measures how fast, and *in which direction*, you're going.

For example, consider a ball thrown straight up into the air. Suppose the ball has velocity (= rate of change of height with respect to time) given by $v(t) = -32t + 72$, where t denotes the number of seconds elapsed from the time the ball is released, and $v(t)$ is measured in feet per second. Then if $v(t)$ is positive, the ball is moving upwards (because $v(t) > 0$ means the height of the ball above the ground is *increasing*), while if $v(t)$ is negative, the ball is moving downwards (since then the height of the ball above the ground is *decreasing*).

Suppose that, the moment the ball is released, it is at a height of 7 feet.

(a) What is the height h of the ball above the ground after 3 seconds? After 6 seconds? After T seconds? Hint: the integral of $v(t)$ measures *change* in height. To get actual height $h(T)$ from this, you need to add the initial height of 7 feet.

(b) At what time t does the ball's velocity equal zero? What does this signify in terms of the ball's motion?

(c) What's the maximum height attained by the ball?

(d) How long does the ball take to reach the ground? Hint: Solve $h(T) = 0$ for T. (Use the quadratic formula; it gives you two solutions. Which one can you ignore?)

8. You have a large circular kiddie pool in your backyard. It has radius 10 feet, so that the area of the base of the pool is $\pi \cdot 10^2 = 100\pi$ square feet. The pool is 8 feet deep. You are using a hose from your house to fill it to the top. **Please supply the appropriate units with all of your answers below.**

(a) (Note: you should be able to do part (a) of the exercise without any calculus.) Your hose supplies 10π cubic feet of water an hour (it's one of those fancy new π-hoses from late night TV).

(i) What will the *depth* of the water be after 1 hour? after 2 hours?

(ii) What will the depth of the water be after T hours?

(iii) How long will it take to fill the pool to 8 feet?

(b) (Note: you probably **do** need calculus for part (b) of this exercise.) This is taking too long, so you buy the new Super Hose (only $19.95 plus shipping and handling, with a free Ginsu Knife if you order before midnight tonight!), which supplies water at rate given by $w(t) = 10\pi t$ cubic feet of water per hour in hour t. (The longer the hose it on, the faster the water flows out of it!)

(i) What will the *volume* of water in the pool be after T hours? Your answer should be a function of T, call it $V(T)$.

(ii) What will the *depth* of the water in the pool be after T hours? Your answer should be a function of T, call it $D(T)$.

(iii) What will the depth of the water be after 1 hour? after 2 hours?

(iv) How long will it take to fill the pool to 8 feet?

(v) What is the rate of change of $D(T)$?