4.2 Accumulation functions

Suppose the change in some quantity E equals some other function p times elapsed time, as long as p is constant during that time. Then we call E an **accumulation function** for p. The formal definition is as follows.

Definition 4.2.1. The function $E(t)$ is an accumulation function for the function $p(t)$ if

$$\Delta E = p(t)\Delta t \tag{4.2.1}$$

over any interval of t-values where $p(t)$ is constant. Here, t is any point in the interval in question. (Since $p(t)$ is constant on this interval, equation (4.2.1) does not depend on which point t in that interval we choose.)

Here is a short table of accumulation functions. (Can you think of others?) The first row in this table was studied in the previous section.

E	p	t
energy	power	time
distance	speed	time
work	force	distance
mass	density	length
force	pressure	area
area	height	length
number of Ebola deaths	death rate	time

Note that the independent variable need not be time.

In Part D of the previous section, we remarked that accumulation functions, derivatives, and areas were all related, at least in the context of power and energy. Here, we wish to elaborate on those relationships, in the more general context of accumulation functions as defined above.

Part A: Accumulation functions and derivatives

If the equality (4.2.1) is true on any interval where $p(t)$ is constant, then we would expect an analogous *approximate equality* to hold on any intervals where $p(t)$ is "approximately constant," meaning $p(t)$ does not vary "too much." In other words: we'd expect that, on such an interval,

$$\Delta E \approx p(t)\Delta t. \tag{4.2.2}$$

Dividing both sides by Δt gives

$$p(t) \approx \frac{\Delta E}{\Delta t}; \tag{4.2.3}$$

letting $\Delta t \to 0$ then gives

$$p(t) = \lim_{\Delta t \to 0} \frac{\Delta E}{\Delta t} = \frac{dE}{dt}. \tag{4.2.4}$$

(The "≈" becomes "=" in the limit because, the smaller Δt is, the less E should vary on the interval in question, so the better the approximation should be.) This is the same argument as was given in Part D of the previous section. But in the present case, we assume only that E is any accumulation function for p, not necessarily that E is energy and p is power.

As remarked previously, the above argument is not entirely rigorous. But it can be made so for functions E and p that are "nice" enough. For example, it suffices that E be locally linear at all points t in question.

We summarize:

If $E(t)$ is an accumulation function for $p(t)$, then $E'(t) = p(t)$, at all points t where $E'(t)$ exists

Accumulation and differentiation

For example, since distance traveled is an accumulation function for speed, we conclude that speed is the derivative of distance traveled. This is not a new result, but rather, a new framework for an old one.

Part B: Accumulation functions and area

Let $E(t)$ be an accumulation function for $p(t)$ on an interval $[a,b]$. For the moment, we'll assume that $p(t)$ is ≥ 0 on $[a,b]$, which is to say that the graph of $p(t)$ never dips below the t-axis on this interval. Later on, we'll relax this assumption.

Here's a possible graph of such a function $p(t)$.

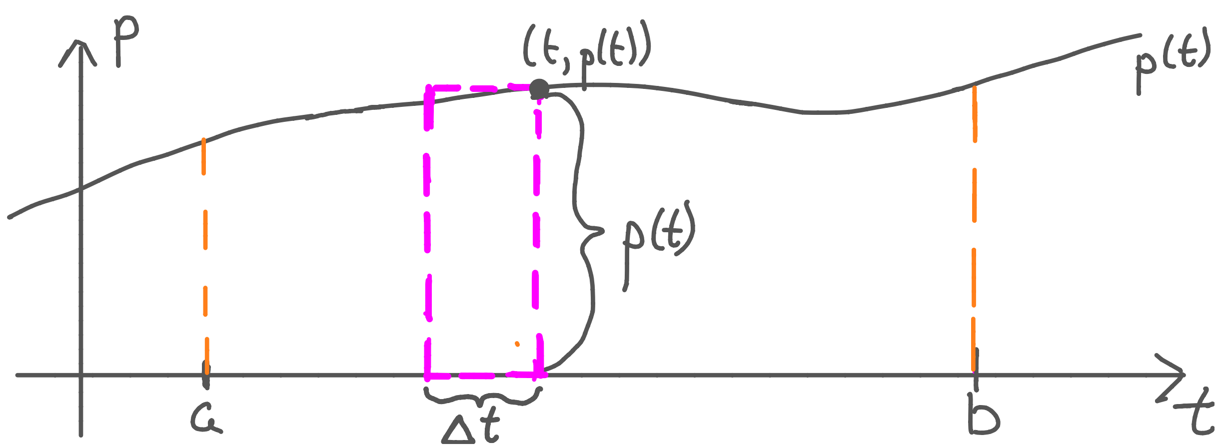


Figure 4.6. A function $p(t)$ and a "representative rectangle"

Consider the dashed rectangle shown in the above figure. Its baselength is Δt and its height is $p(t)$, where t is, in this case, the right endpoint of the given interval of length Δt. The area of this rectangle is therefore equal to height × baselength = $p(t)\Delta t$. That is, by (4.2.2), the area of this rectangle is approximately equal to the change ΔE in the accumulation function $E(t)$, over this interval.

Now imagine that we've spanned the entire interval $[a,b]$ with subintervals like the one above, and have drawn a rectangle like the one above over each of these subintervals. The area of each rectangle approximates, as we've just seen, the change ΔE in the accumulation function over the appropriate interval. So the *sum* of the areas of the rectangles approximates the *sum* of these changes in E. But the sum of the changes in E over the subintervals equals the *overall* change in E over all of $[a,b]$. Conclusion: the change in E over $[a,b]$ is approximately given by the sum of the areas of these rectangles.

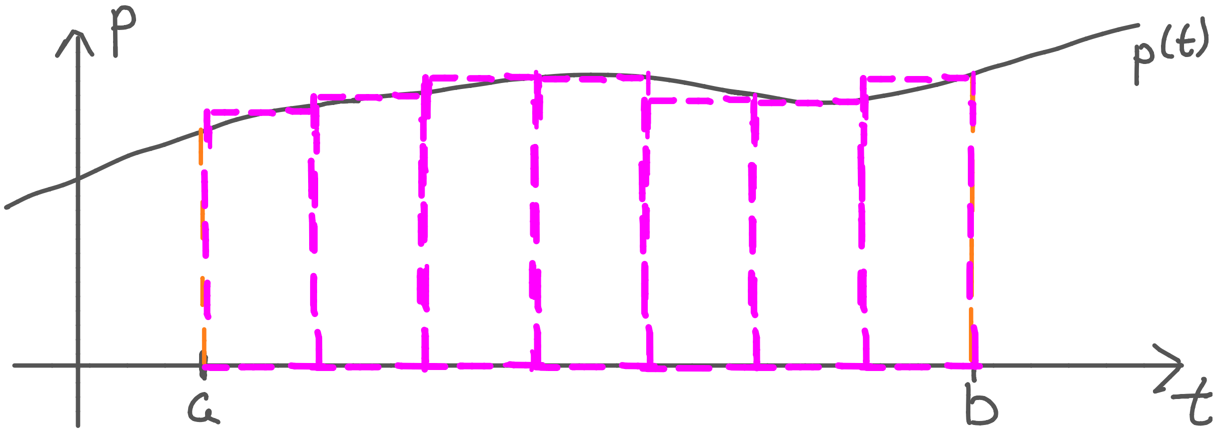


Figure 4.7. The sum of the areas of the rectangles approximates the change in E over $[a,b]$

Now what happens as we "refine" Figure 4.7: that is, we take more and more rectangles, all of which have narrower and narrower baselengths? Well on the one hand, the area of each rectangle should give us a better approximation to the change in E on the interval where that rectangle "lives." So the sum of these areas should give us a better approximation to the change in E over $[a,b]$. On the other hand, the narrower the rectangles, the better they "fit" under the graph of $p(t)$, so the narrower *all* the rectangles are, the closer the *sum* of their areas is to the actual **area under the graph of $p(t)$, over $[a,b]$**.

To summarize the arguments so far: if $E(t)$ is an accumulation function for $p(t)$, over an interval $[a,b]$, and $p(t) \geq 0$ on that interval, then:

$$
\begin{aligned}
&\text{change in } E \text{ over } [a,b] \\
&= \text{sum of changes in } E \text{ over subintervals into which } [a,b] \text{ is subdivided} \\
&\approx \text{sum of terms of the form } p(t)\Delta t\text{, like the one shown in Figure 4.6} \\
&= \text{sum, over all the subintervals, of areas of rectangles like the ones shown in Figure 4.7} \\
&\approx \text{the area under the graph of } p(t)\text{, over } [a,b]. \qquad (4.2.5)
\end{aligned}
$$

There are two separate types of approximations going on here: (i) the change in E over a subinterval is only *approximately* equal to the product $p(t)\Delta t$ there (because $p(t)$ is not actually constant on such a subinterval); and (ii) the areas of all the rectangles, added together, is only *approximately* equal to the area under the graph (because the "space" filled up by the rectangles is not exactly the same as the "space" under the graph). But both of these approximations improve as we take more and more, narrower and narrower, rectangles. So as we do this, all of the "$\approx$" symbols in (4.2.5) become "=," which means that the far left-hand and right-hand sides must, in the end, be *equal.*

We therefore have the following conclusion.

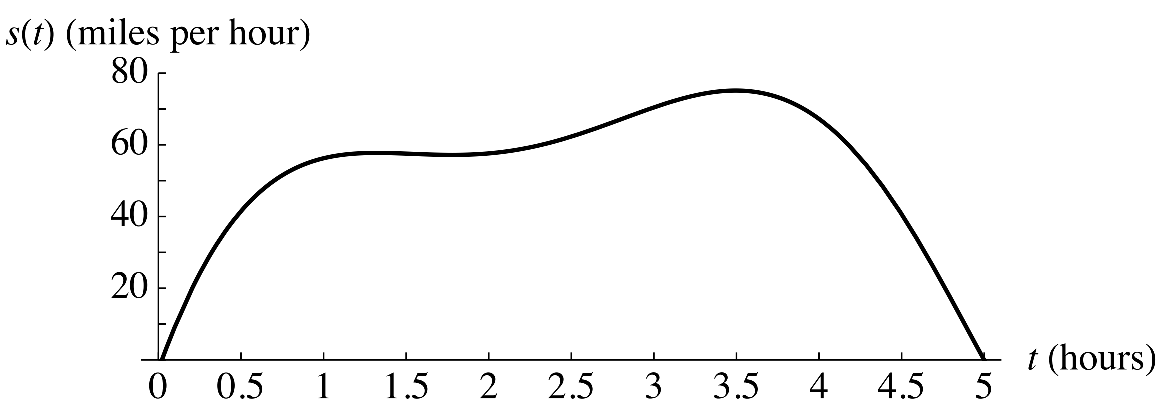
If $E(t)$ is an accumulation function for $p(t)$, and $p(t) \geq 0$ on $[a,b]$, then the change in E over $[a,b]$ equals the area under the graph of $p(t)$, over $[a,b]$

Accumulation and area

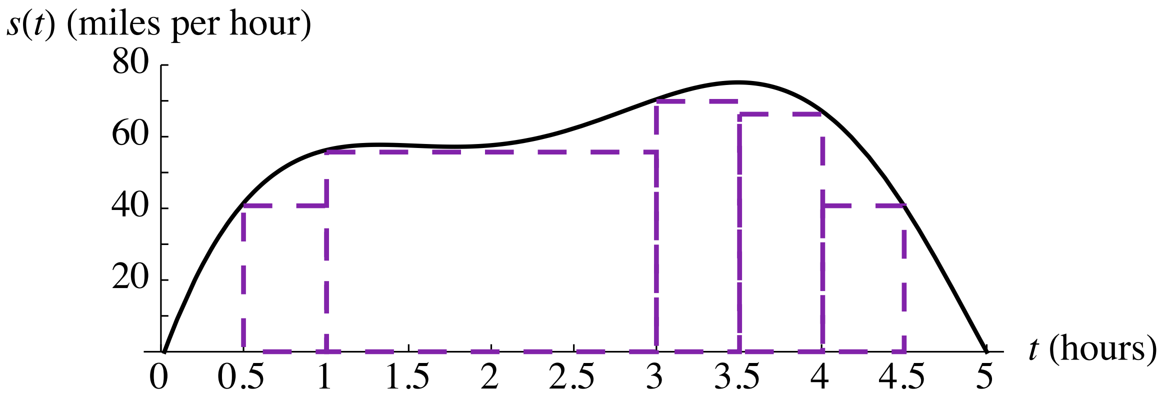
As mentioned earlier, we will soon drop the requirement that $p(t) \geq 0$ on $[a,b]$. Of course, in such a case the graph of $p(t)$ lies *under* the interval $[a,b]$, so we'll have to reinterpret the phrase "over $[a,b]$" in the above conclusion. See Section 4.4.

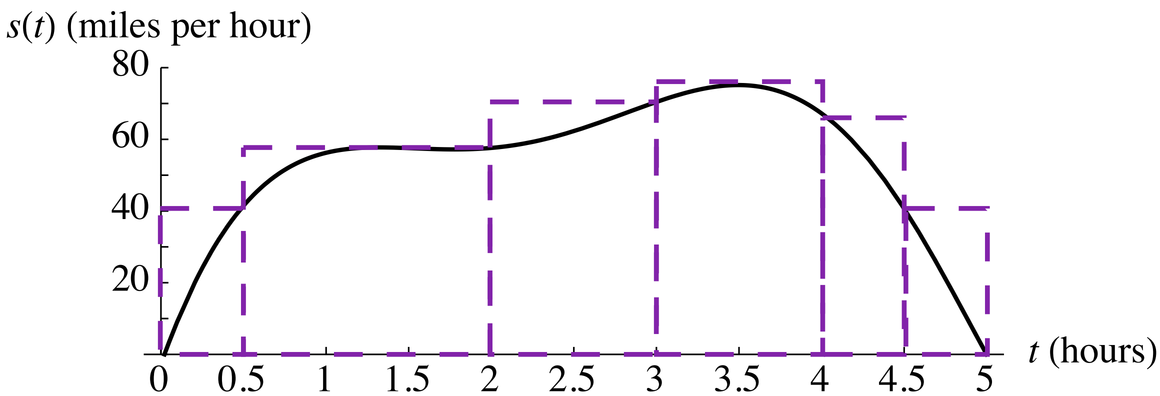
And again, the arguments that brought us to this conclusion are somewhat informal, but they can be made rigorous (so that the conclusion really does hold) as long as neither $p(t)$ nor $E(t)$ is too "weird."

Example 4.2.1. During the course of a certain five-hour journey, a car travels with speed $s(t)$ given by the graph below. Estimate, to within 50 miles, the distance traveled by this car on this trip.



Solution. By the above conclusion, the distance traveled equals the area under the given graph. To approximate this area, consider the two figures below.





In the top figure, we have rectangles whose areas add up to about

$$40 \times 0.5 + 55 \times 2 + 70 \times 0.5 + 65 \times 0.5 + 40 \times 0.5 = 217.5,$$

and these rectangles clearly fill up *less* than the total area under the graph. So the area under the graph must be more than 217.5. In the bottom figure, we have rectangles whose areas add up to about

$$40 \times 0.5 + 60 \times 1.5 + 70 \times 1 + 75 \times 1 + 65 \times 0.5 + 40 \times 0.5 = 307.5,$$

and these rectangles clearly fill up *more* than the total area under the graph. So the area under the graph must be less than 307.5.

The above two approximations are 90 miles apart, so we can't use either as our final estimate, if we want to guarantee that we are no more than 50 miles off. However, note that the average of these two approximations is $(217.5 + 307.5)/2 = 262.5$. So we can estimate that our car traveled 262.5 miles. Since neither 217.5 nor 307.5 is more than 50 miles from 262.5, our estimate should be within 50 miles of the actual distance traveled.

This assumes, of course, that we were fairly accurate in reading values off of the vertical axis of our graph. It turns out that the actual area under the graph shown above is about 268, so our estimate is, in fact, pretty good.

Exercises

Part 1: Work as force × distance

The effort it takes to move an object is called **work**. Since it takes *twice* as much effort to move the object twice as far, or to move another object that is twice as heavy, we can see that the work done in moving an object is proportional to both the force applied and to the distance moved. The simplest way to express this fact is to define

$$\text{work} = \text{force} \times \text{distance}. \tag{4.2.6}$$

For example, to lift a weight of 20 pounds straight up it takes 20 pounds of force. If the vertical distance is 3 feet then

$$20 \text{ pounds} \times 3 \text{ feet} = 60 \text{ foot-pounds}$$

of work is done. (The *foot-pound* is one of the standard units for measuring work.)

The above equation (4.2.6) holds provided the applied force is *constant* over the entire distance in question. If not, then we treat work as an *accumulation function* for force. We can thereby make approximations to amounts of work done, using methods like those employed in Section 4.1.

1. Suppose a tractor pulls a loaded wagon over a road whose steepness varies. If the first 150 feet of road are relatively level and the tractor has to exert only 200 pounds of force while the next 400 feet are inclined and the tractor has to exert 550 pounds of force, how much work does the tractor do altogether?

2. A motor on a large ship is lifting a 2000 pound anchor that is already out of the water at the end of a 30 foot chain. The chain weighs 40 pounds per foot. As the motor lifts the anchor, the part of the chain that is hanging gets shorter and shorter, thereby reducing the weight the motor must lift.

(a) What is the combined weight of anchor and hanging chain when the anchor has been lifted x feet above its initial position? Hint: when the anchor is x feet above its initial position, there are $30 - x$ feet of chain still hanging.

(b) Divide the 30-foot distance that the anchor must move into 3 equal intervals of 10 feet each. Estimate how much work the motor does lifting the anchor and chain over each 10-foot interval by multiplying the combined weight at the *bottom* of the interval by the 10-foot height. (That is: you'll be considering the weight of the chain at $x = 0$, $x = 10$, and $x = 20$.) What is your estimate for the total work done by the motor in raising the anchor and chain 30 feet?

(c) Repeat part (b) of this exercise, but this time, estimate work over each 10-foot interval by multiplying the combined weight at the *top* of the interval by the 10-foot height. (That is: you'll be considering the weight of the chain at $x = 10$, $x = 20$, and $x = 30$.) What is your new estimate for the total work done in raising the anchor and chain 30 feet?

(d) Average your answers from parts (b) and (c) of this exercise, to obtain a better estimate of the amount of work done.

(e) Repeat all the steps of part (b), but this time use 30 equal intervals of 1 foot each. Is your new estimate of the work done larger or smaller than your estimate in part (b)? Which estimate is likely to be more accurate? On what do you base your judgment?

(f) If you ignore the weight of the chain entirely, what is your estimate of the work done? How much *extra* work do you therefore estimate the motor must do to raise the heavy chain along with the anchor?

Part 2: Human work as number of staff × hours

It is natural to say that 6 persons working two hours do the same amount of work as 4 persons working three hours. This suggests that we use the formula

$$\text{human work done} = \text{number of staff working} \times \text{elapsed time} \tag{4.2.7}$$

to measure **human work**. In this situation, if the number of staff working is measured in persons, and elapsed time in hours, then the units of human work are "person-hours."

The above equation (4.2.7) holds provided there is a *constant* number of staff working over the entire distance in question. If not, then we treat human work as an *accumulation function* for number of staff working. We can thereby make approximations to amounts of human work done, using methods like those employed in Section 4.1.

3. House-painting is a job that can be done by several people working simultaneously, Consider a house-painting business run by some students. Because of class schedules, different numbers of students will be painting at different times of the day. Let $S(T)$ be the number of staff present at time T, measured in hours from 8 am, and suppose that during an 8-hour work day, we have

$$S(T) = \begin{cases} 3, & 0 \le T < 2, \\ 2, & 2 \le T < 4.5, \\ 4, & 4.5 \le T \le 8. \end{cases}$$

(a) Draw the graph of the step function defined here, and compute the total number of person-hours of human work done.

(b) Draw the graph that shows how human work *accumulates* on this job. This is the graph of the **accumulated work** function $W(T)$.

(c) Determine the derivative $W'(T)$. Is $W'(T) = S(T)$? If there are points at which this equation fails, identify those points.

4. Suppose that there is a house-painting job to be done, and by past experience the students know that four of them could finish it in 6 hours. But for the first 3.5 hours, only two students can show up, and after that, five will be available.

(a) How long will the whole job take?

(b) Draw a graph of the staffing function for this problem. Mark on the graph the time that the job is finished.

(c) Draw the graph of the accumulated work function $W(T)$.

(d) Determine the derivative $W'(T)$. Is $W'(T) = S(T)$? If there are points at which this equation fails, identify those points.

Average staffing. Suppose a job can be done in three hours when 6 people work the first hour and 9 work during the last two hours. Then the job takes 24 person-hours of work, and the **average staffing** is

$$\text{average staffing} = \frac{24 \text{ person-hours}}{3 \text{ hours}} = 8 \text{ persons.}$$

This means that a *constant* staffing level of 8 persons can accomplish the job in the same time that the given variable staffing level did. Note that the average staffing level (8 persons) is *not* the average of the two numbers 9 and 6!

5. What is the average staffing of the jobs considered in Exercises 3 and 4, above?

6. (a) Draw the graph that shows how work would accumulate in the job described in Exercise 1, if the workforce were kept at the *average* staffing level instead of the varying level described in that exercise. Compare this graph to the graph you drew in Exercise 3(b).

(b) What is the derivative $W'(T)$ of the work accumulation function whose graph you drew in part (a) of this exercise?