3.4 Initial value problems: a review/preview

Recall that, by **initial value problem**, we mean:

(a) One or more **differential equations** (also known as **rate equations**), describing the rates of change of the phenomena of interest, together with

(b) One or more **initial conditions**, specifying values of the dependent variables "at the outset" – that is, at some particular value $t = t_0$ of the independent variable.

The study of initial value problems has been, and will continue to be, a major focus of this book, and of calculus itself. Two substantial questions (among others) in this study are: (i) when, and how, can we **model** a real-world phenomenon by an initial value problem; and (ii) when we do have such a model, how do we **solve** it?

The first question will be addressed in subsequent sections of this chapter, primarily through examples (from which we will extract some fundamental ideas, and some sense of the big picture). In the meantime, in the present section, we review approaches that we've seen so far, to distinguish two general methods of solution.

Let's recall the two initial value problems that we have considered most closely, to this point: **the *SIR* problem**, given by differential equations

$$\begin{aligned} S' &= -aSI \\ I' &= \quad aSI - bI \\ R' &= \qquad\qquad bI \end{aligned} \tag{3.4.1}$$

and initial conditions of the form

$$S(0) = S_0, \quad I(0) = I_0, \quad R(0) = R_0 \tag{3.4.2}$$

(see Sections 1.2 and 1.3); and **the exponential growth/decay problem**, given by a differential equation

$$\frac{dy}{dt} = ky \quad \text{or} \quad \frac{dy}{dt} = -ky \tag{3.4.3}$$

and an initial condition

$$y(0) = C \tag{3.4.4}$$

(see Section 3.1).

For the second type of problem, we found a *closed-form* solution. That is, we found a *direct* equation expressing the sought-after function y in terms of the independent variable t – namely, we found that

$$y = Ce^{kt} \quad \text{(for exponential growth)}, \quad \text{or} \quad y = Ce^{-kt} \quad \text{(for exponential decay)}.$$

Closed-form solutions to initial value problems are quite useful, in that they give us very explicit information, which can, in principle, be studied in detail locally (meaning at or near a given point), as well as globally (meaning over large ranges of the independent variable). We'll consider methods for finding closed-form solutions, when possible, later in this chapter.

In the case of the SIR initial value problem, though, we *did not* find a closed-form solution. That is, we did *not* obtain equations of the form

$$\begin{aligned} S(t) &= \text{some expression involving known functions of } t, \\ I(t) &= \text{some other expression involving known functions of } t, \\ R(t) &= \text{some third expression involving known functions of } t. \end{aligned}$$

Instead we used Euler's method, meaning the idea that

$$\text{new } S = \text{old } S + \Delta S \approx \text{old S} + S'\Delta t,$$

together with the initial values S_0, I_0, and R_0 at time $t = 0$, to approximate S at an instant Δt units later. (Of course, our approximation depended on our choice of Δt.) Similarly, we approximated $I(\Delta t)$ and $R(\Delta t)$. Next, we applied Euler's method to our new (approximate) values at time Δt, to obtain estimates at time $2\Delta t$, and so on. As a result, instead of closed-form formulas for $S(t)$, $I(t)$, and $R(t)$, we obtained *lists*

$$\begin{aligned} &S(0), S(\Delta t), S(2\Delta t), S(3\Delta t), \ldots \\ &I(0), I(\Delta t), I(2\Delta t), I(3\Delta t), \ldots \\ &R(0), R(\Delta t), R(2\Delta t), R(3\Delta t), \ldots. \end{aligned}$$

of successive (approximate) values of S, I, and R.

No closed-form solution to the SIR problem is known to exist. This situation is perhaps more typical of real life (though perhaps not of most calculus classes). That is, the class of initial value problem for which closed-form solutions are tractable is relatively small. Most of the time, one must instead use Euler's method (and/or similar iterative, approximate procedures).

Generalizing the approach we applied to the SIR system, as summarized in Figure 1.1, we have the following "flowchart" for applying Euler's method to initial value problems.

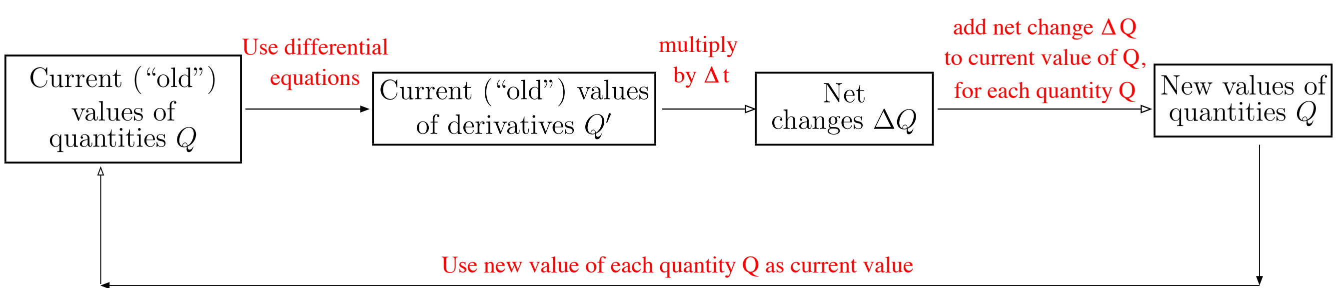

Figure 3.2. Flowchart for Euler's method applied to initial value problems

Of course, the above "loop" has to start somewhere – namely, with the initial conditions, which will tell us our beginning "current" or "old" values. And it has to end somewhere too – this will be determined by the breadth and depth of our inquiry. For example if we start at time 0 and want to predict all the way out to time T, using stepsize Δt, then we'll have to run through our loop

$T/\Delta t$ times, which will give us $(T/\Delta t) + 1$ values of each of our dependent variables (including the initial values).

To summarize: Initial value problems can be used to model many natural phenomena. We can solve such problems in one of two ways:

(a) In closed form, meaning we obtain an equation of the form

$$Q(t) = \text{ some explicit expression in the variable } t$$

for each of the quantities Q in question, or

(b) Numerically, meaning we use Euler's method or a similar one to obtain a list

$$Q(t_0), Q(t_0 + \Delta t),\ Q(t_0 + 2\Delta t), Q(t_0 + 3\Delta t), \ldots$$

for each of the quantities Q in question (where t_0 is the instant at which we begin).

Note that the two methods are not mutually exclusive. It is true, as noted above, that often only the second method is available. However, when closed-form solutions do exist, it is still possible, though typically not preferable, to use Euler's method.