2.3 A global view

Derivative as Function

Up to now we have looked upon the derivative as a *number*. It gives us information about a function at a *point* – the rate at which the function is changing at a point, the slope of the function's graph at a point, and so on.

But the numerical value of the derivative varies from point to point, and these values can also be considered as the values of a new function – the derivative function – with its own graph. Viewed this way the derivative is a *global* object.

The connection between a function and its derivative can be seen very clearly if we look at their graphs. To illustrate, we'll use the function $I(t)$ that describes how the size of an infected population varies over time, from the SIR problem we analyzed in chapter 1. The graph of I appears below, and directly beneath it is the graph of I', the derivative of I. The graphs are lined up vertically: for each t-value a, the values of $I(a)$ and $I'(a)$ are recorded on the same vertical line that passes through the point $t = a$ on the t-axis.

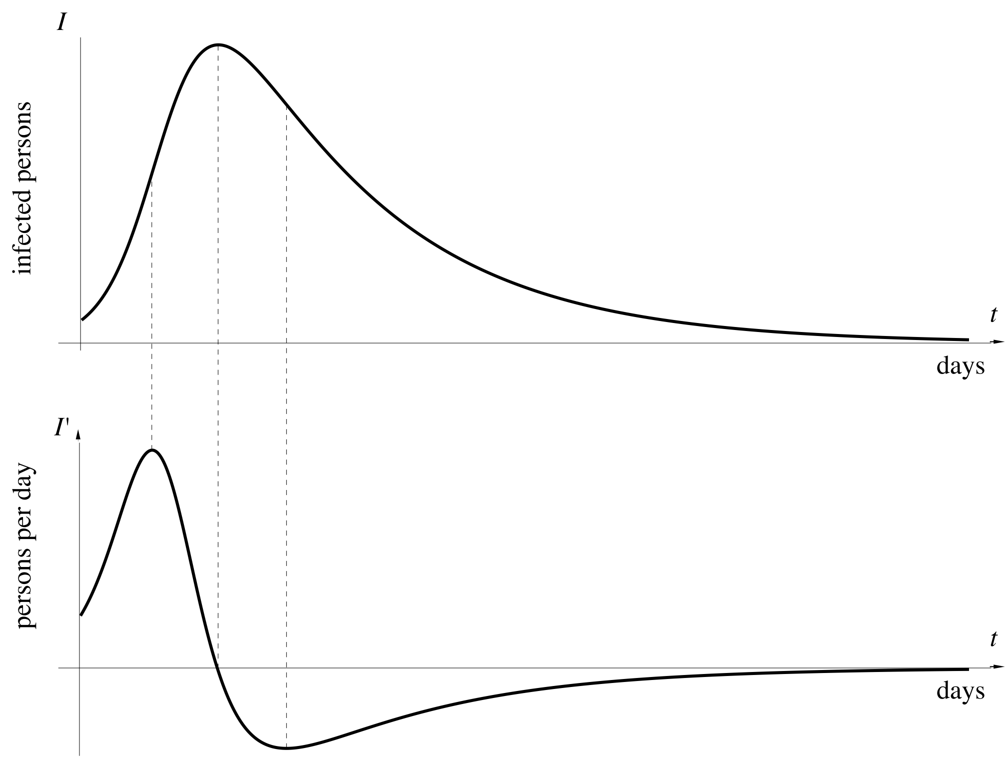


Figure 2.7. The graphs of a function I (top) and of its derivative I' (bottom)

To understand the connection between the graphs, keep in mind that the derivative represents a slope. Thus, at any point t, the *height* of the lower graph (I') tells us the *slope* of the upper graph (I). At the points where I is increasing, I' is positive – that is, I' lies *above* its t-axis. At the point where I is increasing most rapidly, I' reaches its highest value. In other words, where the graph of I is steepest, the graph of I' is highest. At the point where I is decreasing most rapidly, I' has its lowest value.

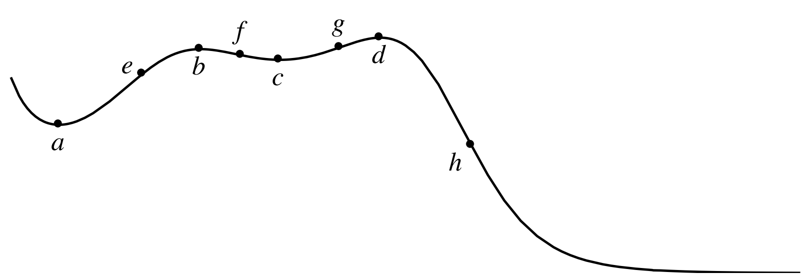
Next, consider what happens when I itself reaches its maximum value. Since I is about to switch from increasing to decreasing, the derivative must be about to switch from positive to negative. Thus, at the moment when I is largest, I' must be zero. Note that the highest point on the graph of I lines up with the point where I' crosses the t-axis. Furthermore, if we zoomed in on the graph of I at its highest point, we would find a horizontal line – in other words, one whose slope is zero.

All functions and their derivatives are related the same way that I and I' are. In the following table we list the various features of the graph of a function; alongside each is the corresponding feature of the graph of the derivative.

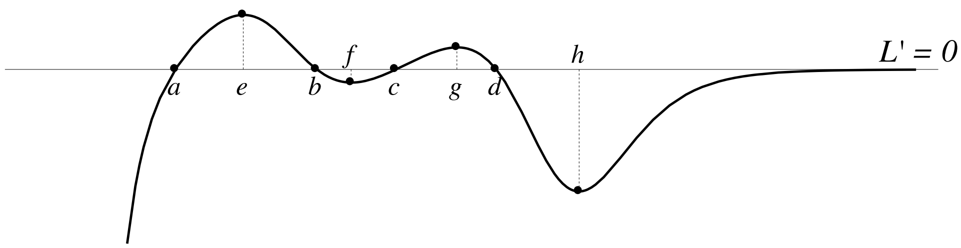
function	derivative
increasing	positive
decreasing	negative
horizontal	zero
steep (rising or falling)	large (positive or negative)
gradual (rising or falling)	small (positive or negative)
straight	horizontal

By using this table, you should be able to make a rough sketch of the graph of the derivative, when you are given the graph of a function. You can also read the table from right to left, to see how the graph of a function is influenced by the graph of its derivative.

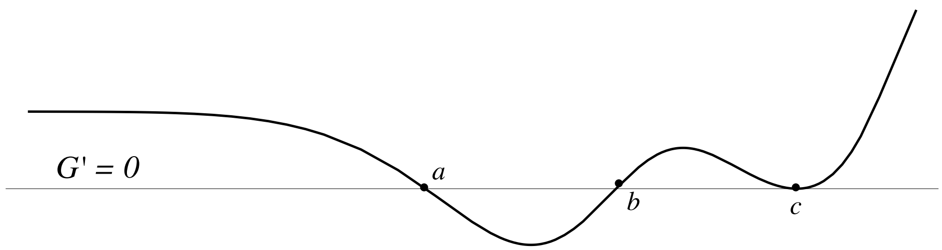
For instance, suppose the graph of the function $L(x)$ is:



Then we know that the derivative L' must be 0 at points a, b, c, and d; that the derivative must be positive between a and b and between c and d, negative otherwise; that the derivative takes on relatively large values at e and g (positive) and at f and h (negative); that the derivative must approach 0 at the right endpoint and be large and negative at the left endpoint. Putting all this together we conclude that the graph of the derivative L' must look something like following:



Conversely, suppose all we are told about a certain function G is that the graph of its derivative G' looks like this:



Then we can infer that the function G itself is decreasing between a and b and is increasing everywhere else; that the graph of G is horizontal at a, b, and c; that both ends of the graph of G slope upward from left to right – the left end more or less straight, the right getting steeper and steeper.

Formulas for Derivatives

The process of finding derivatives of functions is called **differentiation.**

Given a formula for a function f we can, at least in theory, differentiate f, at any point x where f is locally linear. Namely, we can apply the formula

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}. \tag{2.3.1}$$

This formula represents the same definition of the derivative as was given in Section 2.1. But in that section, we wrote "a" to denote our input to f'. Here, we are instead using an "x" for our input to f', to emphasize the fact that this input is a *variable.* That is, we now are thinking globally – we are thinking of $f'(x)$ as a *function* of x, just as $f(x)$ is. (Note that the domain of f' might be smaller than that of f, since there may be points at which $f(x)$ is defined, but not locally linear.)

Derivative calculations using equation (2.3.1) can sometimes be tedious, as you may have noticed in the course of working examples and exercises from Section 2.1. Our goal over the next few sections is to expedite the differentiation process, using two types of tools. The first variety of tool is the *differentiation formula*. By this, we mean an equation that tells us how to differentiate a particular function, or family of functions. The idea here is that certain kinds of functions –

power functions, sines, and cosines, for example – arise frequently. If we catalog the derivatives of such functions, we can then simply "look up" (or remember) those derivatives when we need them.

The second type of tool is the *differentiation rule.* By this we mean a general prescription for expressing the derivative of a complex function in terms of the derivatives of simpler "building blocks."

The idea here is that many complex functions are built up from simpler ones. For instance, the function given by the expression

$$3x^7 + 8\sin(x)\sqrt{x} \tag{2.3.2}$$

is built up from the basic functions x^7, $\sin(x)$, and $\sqrt{x}$. In fact, since $\sqrt{x} = x^{1/2}$, we can think of x^7 and $\sqrt{x}$ as two different instances of the general "power function" x^p.

So, if we have differentiation *formulas* that tell us how to differentiate basic things like x^p and $\sin(x)$, together with differentiation *rules* that tell us how to express the derivative of a complex function in terms of simpler constituent parts, then we *will* be able to find derivatives of complicated things like (2.3.2).

Differentiation formulas

Constant functions. Consider the constant function $f(x) = c$, where c is a fixed real number. The graph of this function is a horizontal line (of height c), and such a line has slope 0 at all points x. But the derivative *is* the slope, so we conclude that

The derivative $f'(x)$ of a constant function $f(x) = c$ is zero at all points x.

The constant formula

Power functions. Before proceeding further, we introduce some new notation. Namely, suppose $y = f(x)$ is a function of x. We will sometimes write

$$\frac{d}{dx}[f(x)],$$

pronounced "dee dee x of f of x," to denote the derivative $f'(x)$. This new notation is called the *Leibniz notation* for the derivative. It has the advantage of allowing us to refer to the derivative of a function given by a formula without giving that function an explicit name. For example, the derivative of the function defined by (2.3.2) may be expressed as

$$\frac{d}{dx}[3x^7 + 8\sin(x)\sqrt{x}].$$

And the above statement "The derivative $f'(x)$ of a constant function $f(x) = c$ is zero at all points x" may simply be written

$$\frac{d}{dx}[c] = 0 \text{ for any constant } c.$$

(In writing a derivative formula using Leibniz notation, we will implicitly assume that the formula holds at all points where the function to be differentiated is locally linear.) Of course we are not restricted to x as a variable name; we might consider

$$\frac{d}{dz}\left[\frac{z^3 + 4^z}{1 + \cos(z)}\right],$$

for example.

With this notation in hand, we now ask: what is

$$\frac{d}{dx}[x^p]$$

for an arbitrary real number p? That is, what is the derivative of a *power function*? We first consider some examples.

Example 2.3.1. Find the derivative $\frac{d}{dx}[x^p]$ for each of the following values of p: (a) $p = 1$; (b) $p = 2$; (c) $p = -1$.

Solution (a) The function $f(x) = x^1 = x$ gives a line with slope 1, so $\frac{d}{dx}[x^1] = 1$.

(b) We compute, using formula (2.3.1), that

$$\begin{aligned}\frac{d}{dx}[x^2] &= \lim_{\Delta x \to 0} \frac{(x + \Delta x)^2 - x^2}{\Delta x} = \lim_{\Delta x \to 0} \frac{x^2 + 2x\Delta x + (\Delta x)^2 - x^2}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{2x\Delta x + (\Delta x)^2}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta x(2x + \Delta x)}{\Delta x} = \lim_{\Delta x \to 0}(2x + \Delta x) = 2x.\end{aligned}$$

(You should compare both the computations and the result here to those of Exercise 2.1.2, which investigated this same derivative, but specifically at the point $x = 3$.)

(c) We have

$$\begin{aligned}\frac{d}{dx}\left[x^{-1}\right] &= \frac{d}{dx}\left[\frac{1}{x}\right] = \lim_{\Delta x \to 0} \frac{1}{\Delta x}\left(\frac{1}{x + \Delta x} - \frac{1}{x}\right) = \lim_{\Delta x \to 0} \frac{1}{\Delta x}\left(\frac{x - (x + \Delta x)}{(x + \Delta x)x}\right) \\ &= \lim_{\Delta x \to 0} \frac{1}{\Delta x}\left(\frac{-\Delta x}{(x + \Delta x)x}\right) = \lim_{\Delta x \to 0} \frac{-1}{(x + \Delta x)x} = \frac{-1}{x^2} = -x^{-2}.\end{aligned}$$

(To get the third equality, we found a common denominator for the two fractions inside the large parentheses.)

The above example suggests a pattern, which we encapsulate as follows:

$$\boldsymbol{\frac{d}{dx}[x^p] = px^{p-1}} \textbf{ for any real number } \boldsymbol{p}$$

The power formula

(The power formula is also sometimes called the power *rule*, but we are reserving the term "rule" for a different context; see the subsection "Differentiation rules," below.)

There is one function to which both the constant and the power formula apply: the function $f(x) = 1 = x^0$. Note that the constant formula tells us that, for this function, $f'(x) = 0$, while the power formula tells us that $f'(x) = \frac{d}{dx}[x^0] = 0x^{0-1} = 0$. That is, the two formulas give the same answer in this case, as they had better! (Technically, neither x^0 nor $0x^{-1}$ is well-defined if $x = 0$. We'll adopt the *convention* that both of these quantities equal 1 in the present context, so that the desired equality holds even at $x = 0$.)

We have by no means proved the power formula for all possible powers p. Note that there are many types of exponents to consider: positive and negative integers; positive and negative rational numbers (that is, fractions), *irrational* powers like $\sqrt{2}$ and π, and so on. Other references will supply proofs for some or all of these kinds of exponents. We will content ourselves, here, with simply *stating* the formula, and reassuring ourselves with the few special cases where we've actually done the computations. And if you are still skeptical, the exercises below explore a few more special cases, for additional reassurance.

In the following example, we investigate another family of functions, where the variable is in the *exponent* – rather than being in the *base*, as it is for the power functions above.

Example 2.3.2. Exponential functions. Let b be a positive constant. Show that

$$\frac{d}{dx}[b^x] = \ln(b)b^x, \tag{2.3.3}$$

where $\ln(b)$, called **the natural logarithm of $\boldsymbol{b}$**, is defined by the formula

$$\ln(b) = \lim_{\Delta x \to 0} \frac{b^{\Delta x} - 1}{\Delta x}. \tag{2.3.4}$$

Solution. Using the fact that $b^{x+y} = b^x b^y$ for all real numbers x and y, we find that

$$\frac{d}{dx}[b^x] = \lim_{\Delta x \to 0} \frac{b^{x+\Delta x} - b^x}{\Delta x} = \lim_{\Delta x \to 0} \frac{b^x b^{\Delta x} - b^x}{\Delta x} = \lim_{\Delta x \to 0} \frac{b^x(b^{\Delta x} - 1)}{\Delta x}. \tag{2.3.5}$$

Now note that the factor b^x, on the right-hand side of equation (3.5.4), is independent of Δx. So we may, in fact, move this factor out in front of the "$\lim_{\Delta x \to 0}$." We do so using the eminently plausible, and *true*, fact (whose proof we omit) that "the limit of a constant multiple equals the constant multiple of the limit." That is, if $Q(\Delta x)$ is some quantity that approaches a number L as $\Delta x \to 0$, and if c is a number that does not depend on Δx, then $c\,Q(\Delta x)$ approaches cL as $\Delta x \to 0$. This fact, together with (3.5.4), tell us that

$$\frac{d}{dx}[b^x] = b^x \lim_{\Delta x \to 0} \frac{b^{\Delta x} - 1}{\Delta x} = \ln(b)b^x,$$

as required.

A few comments on Example 2.3.2 are in order. First: we require that $b > 0$ to avoid difficulties like square roots of negative numbers. For example, were we to take $b = -1$, then letting $x = 1/2$ would give $b^x = (-1)^{1/2} = \sqrt{-1}$, which is not defined (as a real number). Stipulating that $b > 0$ eliminates such issues.

Second: $\ln(b)$ depends on b, but not on x. So the above example demonstrates the fundamental fact that *the derivative of an exponential function is a constant (with respect to the independent variable) times that function*. We've encountered functions that behave like this before – see the subsection "Proportionality in rate equations," and the associated exercises, in Section 1.5 above. There, we considered quantities P satisfying rate equations of the form $P' = kP$. In light of Example 2.3.2, then, we should expect that such quantities P are exponential in nature. This expectation will be borne out in Section 3.1 below.

Third: the definition (2.3.4) of $\ln(b)$ is a bit complicated. And in general, there's no algebra we can do to extract precise numerical values from this definition – except for the case $b = 1$, since

$$\ln(1) = \lim_{\Delta x \to 0} \frac{1^{\Delta x} - 1}{\Delta x} = \lim_{\Delta x \to 0} \frac{1 - 1}{\Delta x} = \lim_{\Delta x \to 0} \frac{0}{\Delta x} = \lim_{\Delta x \to 0} 0 = 0.$$

For more general $b > 0$, we can nonetheless obtain, from (2.3.4), an arbitrarily good *approximation* to $\ln(b)$. We do so by evaluating the difference quotient in (2.3.4) at a suitably small value of Δx. For example: putting $b = 2$ and $\Delta x = 0.000001$ gives

$$\ln(2) \approx \frac{2^{0.000001} - 1}{0.000001} = 0.69314...,$$

which is correct to five decimal places.

Fourth: you may have encountered natural logarithms in other contexts, where you probably saw them defined differently. We'll see in Chapter 3 that other, perhaps more familiar, definitions of $\ln(b)$ are equivalent to the one given above.

Fifth, and last: note the fundamental difference between a *power function*, where the input, or independent variable, is in the base, and an *exponential function*, where the input appears in the exponent. When you differentiate a power function, the original exponent drops by 1, and also appears as a factor in your derivative. When you differentiate an exponential function, the exponent doesn't change, but the natural logarithm of the *base* appears as a factor. For example,

$$\frac{d}{dx}[x^\pi] = \pi x^{\pi - 1} \quad \text{but} \quad \frac{d}{dx}[\pi^x] = \ln(\pi)\pi^x.$$

The final general category of function that we will consider, in this section, is the category of trigonometric functions. For example, let $h'(x) = \sin(x)$: we saw in Example 2.1.3 above that $h'(0) = 1$. Using ideas from that example, together with some trigonometric identities, we can similarly show that $h'(x) = \cos(x)$ for all x. And we can analogously compute $\frac{d}{dx}[\cos(x)]$ and $\frac{d}{dx}[\tan(x)]$. See the Exercises below. We compile these results, along with the others catalogued above, into the following table.

function $f(x)$	derivative $f'(x) = \frac{d}{dx}[f(x)]$
c	0
x^p	px^{p-1}
b^x	$\ln(b)b^x$
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$
$\tan(x)$	$\sec^2(x)$

Table 2.1 A short table of derivative formulas

Here c and p can be any real numbers, and b can be any positive number. Also, recall that $\sec(x) = 1/\cos(x)$, and that $\sec^2(x)$ is shorthand for $(\sec(x))^2$.

Again, remember that the input to the trigonometric functions is always measured in radians; the above formulas are not correct if x is measured in degrees. There are similar formulas if you insist on using degrees, but they are more complicated. This is the principal reasons we work in radians – the derivative formulas are nice!

Differentiation Rules

Since basic functions are combined in various ways to make formulas, we need to know how to differentiate *combinations*. For example, suppose we add the functions $g(x)$ and $h(x)$, to get $f(x) = g(x) + h(x)$. It may be shown, then, that f is differentiable too, and moreover, that "the rate at which f changes is the sum of the separate rates at which g and h change," or "the derivative of the sum is the sum of the derivatives." More formally, we have

$$\boxed{\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x)}$$

The sum rule

For example,

$$\text{if } f(x) = \tan(x) + x^{-6}, \text{ then } f'(x) = \sec^2(x) - 6x^{-7};$$
$$\frac{d}{dw}\left[2^w + \cos(w)\right] = \ln(2)2^w - \sin(w).$$

Likewise, if we multiply any differentiable function g by a constant c, then, as one can show, the product $f(x) = cg(x)$ is also differentiable, and moreover "the derivative of the constant multiple equals the constant multiple of the derivative," or "rescaling a function rescales its rate of change by the same factor." That is,

$$\boxed{\frac{d}{dx}[cf(x)] = cf'(x)}$$

The constant multiple rule

For example, $\frac{d}{dx}[5\sin(x)] = 5\cos(x)$. Likewise, if $g(z) = 25z^3$, then $g'(z) = 25 \times 3z^2 = 75z^2$. However, the rule does *not* tell us how to find the derivative of $\sin(5x)$, because $\sin(5x) \neq 5\sin(x)$. To work this one out, we will need the **chain rule**, which describes how to differentiate compositions $f(g(x))$, in terms of the derivatives f' and g'. See the following section. Subsequent sections will also investigate derivatives of products $f(x)g(x)$ and quotients $f(x)/g(x)$.

With just the few facts already laid out, we can differentiate a variety of functions given by formulas. Here are a couple more simple examples.

Example 2.3.3. Find:

(a) $\frac{d}{dq}\left[-\frac{-7\cos(q)}{13} + 5\sqrt[3]{q^{11}} + \frac{1}{\sqrt{q}} + \frac{q^5}{4923} + 4923\,5^q - 23944923\pi^\pi\right]$.

(b) The derivative of the general **polynomial** function

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0.$$

Here a_n, a_{n-1}, …, a_2, a_1, a_0 are all constants, and n is a positive integer, called the **degree** of the polynomial.

Solution. (a) We express the second and third summands in terms of powers, so that we can use the power formula. Then, by Table 2.1 and the sum and constant multiple rules, above,

$$\begin{aligned}
&\frac{d}{dq}\left[-\frac{7\cos(q)}{13} + 5\sqrt[3]{q^{11}} + \frac{1}{\sqrt{q}} + \frac{q^5}{4923} + 4923\,5^q - 23944923\pi^\pi\right] \\
&= \frac{d}{dq}\left[-\frac{7\cos(q)}{13} + 5q^{11/3} + q^{-1/2} + \frac{1}{4923}q^5 + 4923\,5^q - 23944923\pi^\pi\right] \\
&= -\frac{7}{13}\frac{d}{dq}[\cos(q)] + 5\frac{d}{dq}[q^{11/3}] + \frac{d}{dq}[q^{-1/2}] + \frac{1}{4923}\frac{d}{dq}[q^5] + 4923\frac{d}{dq}[5^q] - 23944923\frac{d}{dq}[\pi^\pi] \\
&= -\frac{7}{13}(-\sin(q)) + 5 \times \frac{11}{3}q^{8/3} - \frac{1}{2}q^{-3/2} + \frac{1}{4923} \times 5q^4 + 4923 \times \ln(5)5^q - 23944923 \times 0 \\
&= \frac{7}{13}\sin(q) + \frac{55}{3}q^{8/3} - \frac{1}{2}q^{-3/2} + \frac{5q^4}{4923} + 4923\ln(5)5^q.
\end{aligned}$$

(b) The polynomial $P(x)$ is a sum of terms, each of which is a constant multiple of an integer power of the input variable. (A polynomial of degree 1 is just a linear function.) The derivative of $P(x)$, by the sum and constant multiple rues and the power and constant formulas above,

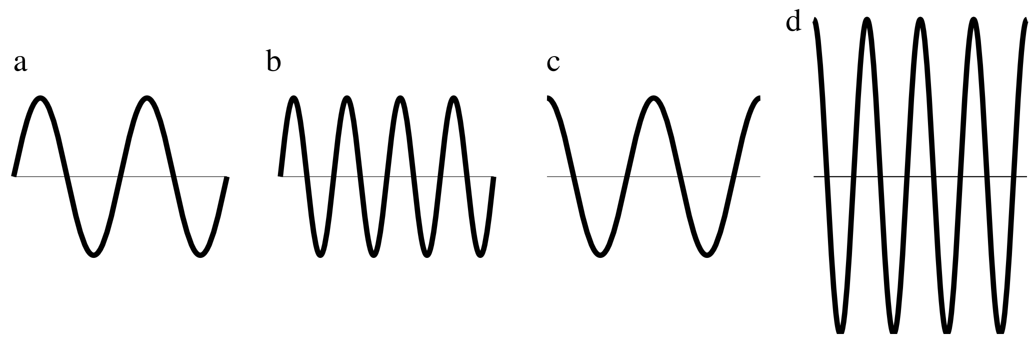
$$P'(x) = na_n x^{n-1} + (n-1)a_{n-1}x^{n-2} + \cdots + 2a_2 x + a_1.$$

Exercises

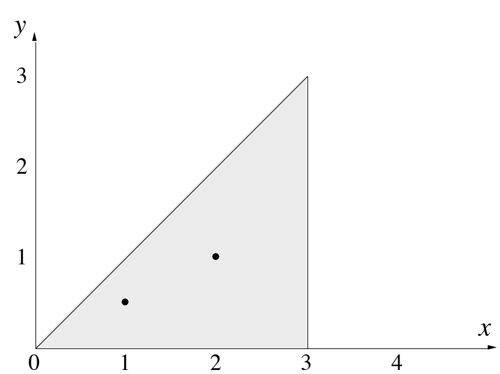
Part 1: Sketching the graph of the derivative

1. Use a computer utility to sketch the graphs of two different *linear* functions that have the same derivative.

2. Here are the graphs of four related functions: s, its derivative s', another function $c(t) = s(2t)$, and *its* derivative $c'(t)$. The graphs are out of order. Label them with the correct names s, s', c, and c'.



3. (a) Suppose a function $y = g(x)$ satisfies $g(0) = 0$ and $0 \leq g'(x) \leq 1$ for all values of x in the interval $0 \leq x \leq 3$. Explain carefully why the graph of g must lie entirely in the triangular region shaded below:



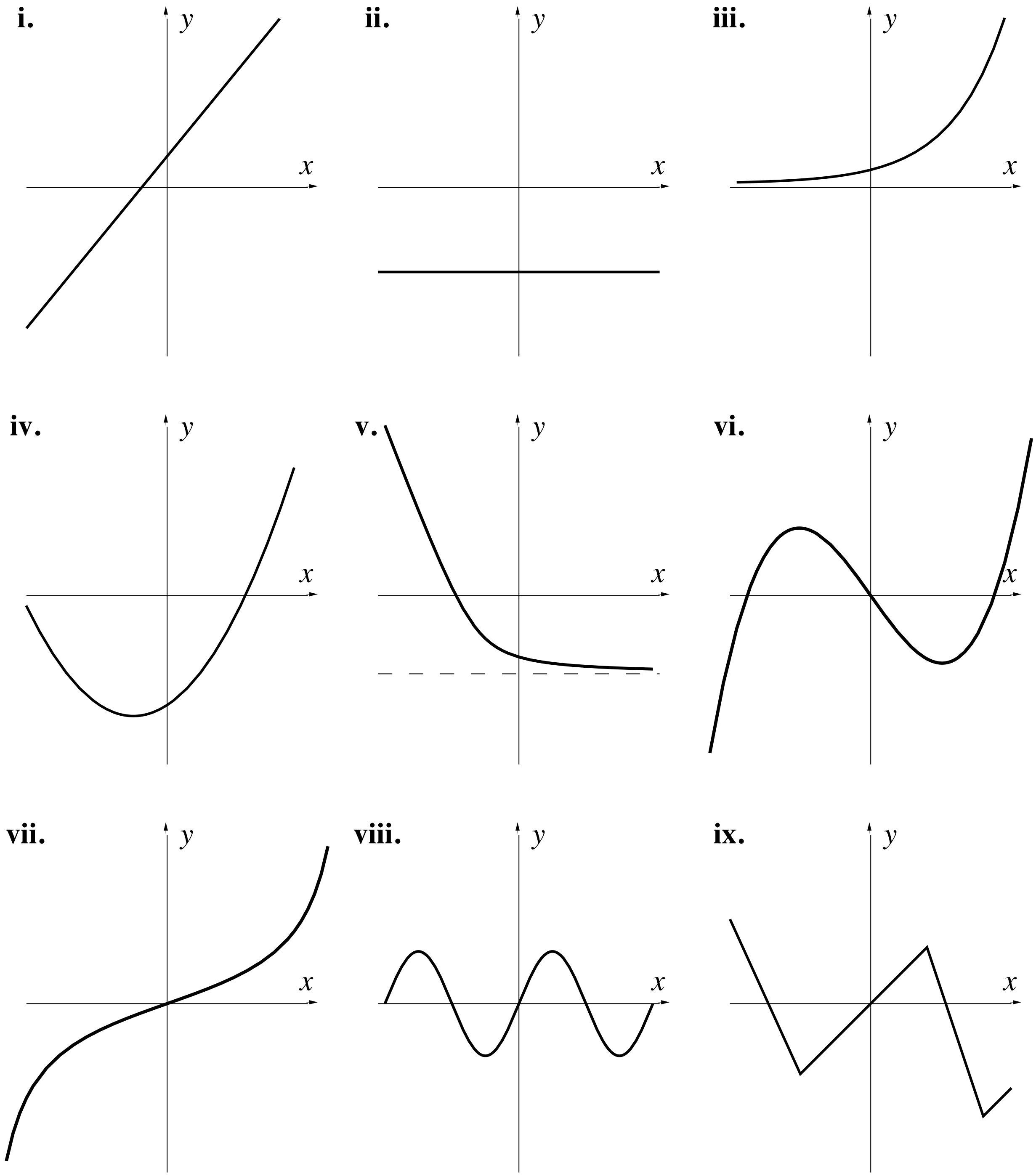
(b) Suppose you learn that $g(1) = .5$ and $g(2) = 1$. Draw the smallest shaded region in which you can guarantee that the graph of g must lie.

4. Suppose h is differentiable over the interval $0 \leq x \leq 3$. Suppose $h(0) = 0$, and that

$$\begin{aligned} .5 \leq h'(x) \leq 1 \quad &\text{for} \quad 0 \leq x \leq 1 \\ 0 \leq h'(x) \leq .5 \quad &\text{for} \quad 1 \leq x \leq 2 \\ -1 \leq h'(x) \leq 0 \quad &\text{for} \quad 2 \leq x \leq 3 \end{aligned}$$

Draw the smallest shaded region in the x, y-plane in which you can guarantee that the graph of $y = h(x)$ must lie.

5. For each of the functions graphed below, sketch the graph of its derivative.



Part 2: Differentiation using the definition of the derivative

For each exercise in this section, use the definition (2.3.1) to deduce the indicated derivative formula.

6. (a) Show that, if $f(x) = \sqrt{x}$, then $f'(x) = \dfrac{1}{2\sqrt{x}}$. For this exercise, you might wish to use the

algebra "trick" of multiplying both numerator and denominator of a fraction by the same thing. More specifically, note that

$$f'(x) = \lim_{\Delta x\to 0} \frac{\sqrt{x+\Delta x}-\sqrt{x}}{\Delta x} = \lim_{\Delta x\to 0} \frac{\sqrt{x+\Delta x}-\sqrt{x}}{\Delta x} \cdot \frac{\sqrt{x+\Delta x}+\sqrt{x}}{\sqrt{x+\Delta x}+\sqrt{x}}$$
$$= \lim_{\Delta x\to 0} \frac{(\sqrt{x+\Delta x}-\sqrt{x})(\sqrt{x+\Delta x}+\sqrt{x})}{\Delta x(\sqrt{x+\Delta x}+\sqrt{x})}.$$

Do the algebra in the numerator; you should end up with a numerator that involves no square roots. (Use the fact that $\left(\sqrt{Y}\right)^2 = Y$ for any Y.) You should then be able to cancel out a Δx in your numerator and denominator. Then compute what happens to what's left, as $\Delta x \to 0$.

(b) Explain why part (a) of this exercise agrees with the case $p = 1/2$ of the power formula.

7. (a) Show that, if $g(x) = \frac{1}{\sqrt{x}}$, then $g'(x) = -\frac{1}{2\sqrt{x^3}}$. Hint: obtaining a common denominator gives

$$g'(x) = \lim_{\Delta x\to 0} \frac{1}{\Delta x}\left(\frac{1}{\sqrt{x+\Delta x}} - \frac{1}{\sqrt{x}}\right) = \lim_{\Delta x\to 0} \frac{1}{\Delta x}\left(\frac{\sqrt{x}-\sqrt{x+\Delta x}}{\sqrt{x(x+\Delta x)}}\right).$$

Multiply the right-hand side by

$$\frac{\sqrt{x}+\sqrt{x+\Delta x}}{\sqrt{x}+\sqrt{x+\Delta x}}$$

and simplify. You should then be able to cancel out a Δx top and bottom, and then take the appropriate limit.

(b) Explain why part (a) of this exercise agrees with the case $p = -1/2$ of the power formula.

8. (a) Show that, if $k(x) = \frac{1}{x^2}$, then $k'(x) = -\frac{2}{x^3}$. Hint obtaining a common denominator gives

$$k'(x) = \lim_{\Delta x\to 0} \frac{1}{\Delta x}\left(\frac{1}{(x+\Delta x)^2} - \frac{1}{x^2}\right) = \lim_{\Delta x\to 0} \frac{1}{\Delta x}\left(\frac{x^2-(x+\Delta x)^2}{x^2(x+\Delta x)^2}\right).$$

Perform some algebra in the numerator; then cancel out a Δx top and bottom; the consider what happens as $\Delta x \to 0$.

(b) Explain why part (a) of this exercise agrees with the case $p = -2$ of the power formula.

9. Show that, if $h(x) = \sin(x)$, then $h'(x) = \cos(x)$. Hint: using the trigonometric identity

$$\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b),$$

we find that

$$h'(x) = \lim_{\Delta x\to 0} \frac{\sin(x+\Delta x)-\sin(x)}{\Delta x} = \lim_{\Delta x\to 0} \frac{\sin(x)\cos(\Delta x)+\cos(x)\sin(\Delta x)-\sin(x)}{\Delta x}.$$

If you collect some terms in the numerator, then you can use the limit formulas

$$\lim_{\Delta x \to 0} \frac{\sin(\Delta x)}{\Delta x} = 1 \quad \text{and} \quad \lim_{\Delta x \to 0} \frac{\cos(\Delta x) - 1}{\Delta x} = 0.$$

10. Show that $\frac{d}{dx}[\cos(x)] = -\sin(x)$. The process is similar to that of Exercise 9 above, except that, in the present case, you'll want to use the trigonometric identity

$$\cos(a + b) = \cos(a)\cos(b) - \sin(a)\sin(b).$$

11. Show that $\frac{d}{dx}[\tan(x)] = \sec^2(x)$. Hint: use the fact, which may be shown using the trigonometric identities given in the preceding two exercises, that

$$\tan(x + \Delta x) - \tan(x) = \frac{\sin(\Delta x)\sec(x)}{\cos(\Delta x)\cos(x) - \sin(\Delta x)\sin(x)}.$$

You may also want to use one of the limit formulas from Exercise 9, and the facts that $\sin(0) = 0$ and $\cos(0) = 1$.

Part 3: Differentiation using rules and formulas

12. Find the indicated derivatives, using rules and formula for this section.

(a) $f'(x)$ if $f(x) = 3x^7 - .3x^4 + \pi x^3 - 17$

(b) $\frac{d}{dx}\left[\sqrt{3}\,\sqrt{x} + \frac{7}{x^5}\right]$

(c) $h'(w)$ if $h(w) = \frac{w^8}{12} - \sin(w) + \frac{1}{3w^2}$

(d) $\frac{d}{du}\left[\frac{4\cos(u)}{5} - \frac{3\tan(u)}{8} + \sqrt[3]{u}\right]$

(e) $V'(s)$ if $V(s) = \sqrt[4]{16} - \sqrt[4]{s}$

(f) The derivative with respect to z of $F(z) = \sqrt{7} \cdot 2^z + (1/2)^z$

(g) $P'(t)$ if $P(t) = -\frac{a}{2}t^2 + v_0 t + d_0$ (a, v_0, and d_0 are constants)

13. Use a computer graphing utility for this exercise.

For each of the parts (a),(b),(c),(d) of this exercise, graph all three of the following functions on the same set of axes:

(i) The given function f, on the indicated interval;

(ii) The function $g(x) = (f(x + .01) - f(x))/0.01$ that estimates the slope of the graph of f at x;

(iii) The function $h(x) = f'(x)$, where you use the differentiation rules to find f'.

(a) $f(x) = x^4$ on $-1 \leq x \leq 1$.

(b) $f(x) = x^{-1}$ on $1 \leq x \leq 8$.

(c) $f(x) = \sqrt{x}$ on $.25 \leq x \leq 9$.

(d) $f(x) = \sin(x)$ on $0 \leq x \leq 2\pi$.

The graphs of g and h should coincide, at least roughly, in each case. Do they?

14. In each case below, find a function $f(x)$ whose derivative $f'(x)$ is:

(a) $f'(x) = 12\,x^{11}$.

(b) $f'(x) = 5x^7$.

(c) $f'(x) = \cos(x) + \sin(x)$.

(d) $f'(x) = ax^2 + bx + c$.

(e) $f'(x) = 0$.

(f) $f'(x) = \dfrac{5}{\sqrt{x}}$.

15. What is the slope of the graph of $y = x - \sqrt{x}$ at $x = 4$? At $x = 100$? At $x = 10000$?

16. (a) For which values of x is the function $x - x^3$ increasing?

(b) Where is the graph of $y = x - x^3$ rising most steeply?

(c) At what points is the graph of $y = x - x^3$ horizontal?

(d) Make a sketch of the graph of $y = x - x^3$ that reflects all these results.

17. (a) Sketch the graph of the function $y = 2x + \dfrac{5}{x}$ on the interval $0.2 \leq x \leq 4$.

(b) Where is the lowest point on that graph? Give the value of the x-coordinate *exactly*. [Answer: $x = \sqrt{5/2}$.]

18. What is the slope of the graph of $y = \sin(x) + \cos(x)$ at $x = \pi/4$?

19. Write down two quadratic polynomials $f(x)$ and $g(x)$ that have the same derivative. (A quadratic polynomial is a polynomial of degree two. See Example 2.3.3 above.) Supply a computer graph of these two functions, both graphed on the same set of axes.

20. A ball is held motionless and then dropped from the top of a 200 foot tall building. After t seconds have passed, the distance from the ground to the ball is $d = f(t) = -16t^2 + 200$ feet.

(a) Find a formula for the velocity $v = f'(t)$ of the ball after t seconds. Check that your formula agrees with the given information that the initial velocity of the ball is 0 feet/second.

(b) Draw graphs of both the velocity and the distance as functions of time. What time interval makes physical sense in this situation? (For example, does $t < 0$ make sense? Does the distance formula make sense after the ball hits the ground?)

(c) At what time does the ball hit the ground? What is its velocity then?

21. A second ball is tossed straight up from the top of the same building with a velocity of 10 feet per second. After t seconds have passed, the distance from the ground to the ball is $d = f(t) = -16t^2 + 10t + 200$ feet.

(a) Find a formula for the velocity of the second ball. Does the formula agree with given information that the initial velocity is +10 feet per second? Compare the velocity formulas for the two balls; how are they similar, and how are they different?

(b) Draw graphs of both the velocity and the distance as functions of time. What time interval makes physical sense in this situation?

(c) Use your graph to answer the following questions. During what period of time is the ball rising? During what period of time is it falling? When does it reach the highest point of its flight?

(d) How high does the ball rise?

22. (a) What is the velocity formula for a third ball that is thrown *downward* from the top of the building with a velocity of 40 feet per second? Check that your formula gives the correct initial velocity.

(b) What is the distance formula for the third ball? Check that it satisfies the initial condition (namely, that the ball starts at the top of the building).

(c) When does this ball hit the ground? How fast is it going then?

23. A steel ball is rolling along a 20-inch long straight track so that its distance from the midpoint of the track (which is 10 inches from either end) is $d = 3 \sin t$ inches after t seconds have passed. (Think of the track as aligned from left to right. Positive distances mean the ball is to the right of the center; negative distances mean it is to the left.)

(a) Find a formula for the velocity of the ball after t seconds. What is happening when the velocity is positive; when it is negative; when it equals zero? Write a sentence or two describing the motion of the ball.

(b) How far from the midpoint of the track does the ball get? How can you tell?

(c) How fast is the ball going when it is at the midpoint of the track? Does it ever go faster than this? How can you tell?

24. A forester who wants to know the height of a tree walks 100 feet from its base, sights to the top of the tree, and finds the resulting angle to be 57 degrees.

(a) What height does this give for the tree?

(b) If the measurement of the angle is certain only to 5 degrees, what can you say about the uncertainty of the height found in part (a)? (Note: you need to express angles in *radians* to use the formulas from calculus: π radians = 180 degrees.)

25. (a) In the preceding problem, what percentage error in the height of the tree is produced by a 1 degree error in measuring the angle?

(b) What would the percentage error have been if the angle had been 75 degrees instead of 57 degrees? 40 degrees?

(c) If you can measure angles to within 1 degree accuracy and you want to measure the height of a tree that's roughly 150 feet tall by means of the technique in the preceding problem, how far away from the tree should you stand to get your best estimate of the tree's height? How accurate would your answer be?