2.2 Local linearity (differentiability)

We begin with the following.

Definition 2.2.1. We say that the function $y = f(x)$ is **locally linear**, or **differentiable**, at the point $x = a$ if the limit

$$\lim_{\Delta x \to 0} \frac{f(a + \Delta x) - f(a)}{\Delta x} \tag{2.2.1}$$

exists.

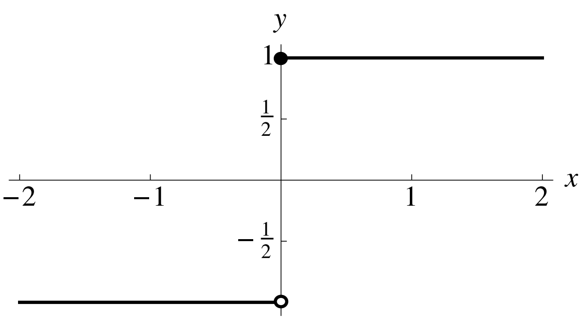
We simply say "f is locally linear" (or "differentiable") if it's locally linear at *all* points in a specified domain.

We saw in the previous section that this limit, if it exists, is the derivative $f'(a)$ of $y = f(x)$ at $x = a$. So: "$y = f(x)$ is locally linear, or differentiable, at the point $x = a$" simply means "the derivative $f'(a)$ exists."

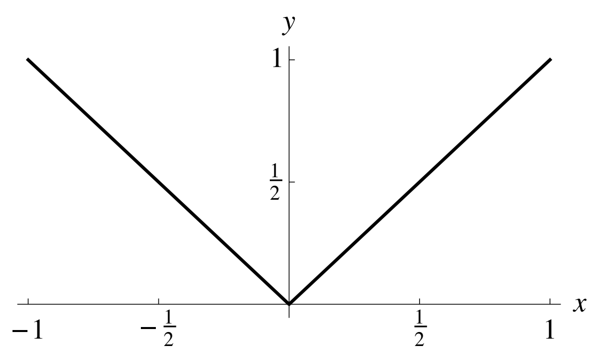
Geometrically, existence of $f'(a)$ means existence of the slope of the line tangent to $y = f(x)$ at $x = a$. To understand what it means for this slope to exist, it is instructive to first consider some situations where it *does not* exist.

Example 2.2.1. Consider the functions

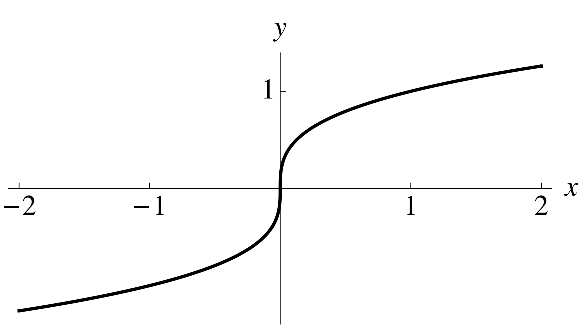
$$k(x) = \operatorname{sgn}(x) = \begin{cases} -1 & \text{if } x < 0, \\ 1 & \text{if } x \geq 0, \end{cases} \qquad f(x) = |x|, \qquad g(x) = \sqrt[3]{x} = x^{1/3}, \qquad h(x) = \sqrt[3]{x^2} = x^{2/3}.$$



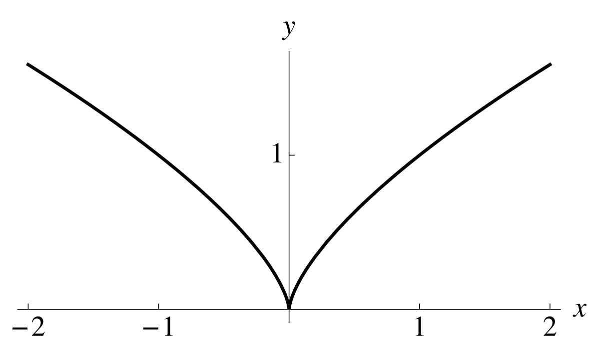
(i) $k(x) = \operatorname{sgn}(x)$: ambiguous tangent at $x = 0$



(ii) $f(x) = |x|$: ambiguous tangent at $x = 0$



(iii) $g(x) = x^{1/3}$: vertical tangent at $x = 0$



(ii) $h(x) = x^{2/3}$: ambiguous/vertical tangent at $x = 0$

Figure 2.5. Some functions with curious behavior $x = 0$

Explain geometrically why each of these functions fails to be differentiable at $x = 0$.

Solution. Because of the "jump" in the graph of $k(x) = \text{sgn}(x)$ at $x = 0$, there's no unambiguous way to define a line "just touching" this graph at $x = 0$. That is, there's no unambiguous definition of a tangent line, and therefore no unambiguous definition of a derivative, at this point.

Regarding $f(x) = |x|$: we might imagine that *many* lines are just touching the graph of this function at $x = 0$. For example, the lines $y = -x/2$, $y = -x/3$, and $y = 3x/4$ all "hinge" on this graph at this point. These lines all have different slopes, so there's no unambiguous definition of $f'(0)$.

The graph of $g(x) = x^{1/3}$ *does* have a unique tangent line at $x = 0$, but this tangent line is vertical, and therefore does not have a finite slope. Therefore, we can not define $g'(0)$.

Finally: as with $y = |x|$, one can imagine many different lines "hinging" on the graph of $h(x) = x^{2/3}$ at $x = 0$. One could even imagine a vertical line balancing on this graph there. So this function is about as bad as one can get, from the point of view of differentiability: not only are there too many possible candidates for $h'(0)$, but one of those candidates is undefined.

We have some sense, then, of how a derivative can go wrong. But now, let's reflect on what it means for a derivative to go *right*. What we wish to argue is that:

If $f'(a)$ exists then, in a vicinity of $x = a$, $f(x)$ looks a lot like its tangent line.

To make this argument, we observe the following. Again, $f'(a)$ is an instantaneous rate of change, while the difference quotient

$$\frac{f(a + \Delta x) - f(a)}{\Delta x}$$

is an average rate of change. And this average rate of change should, intuitively, be pretty close to this instantaneous rate of change if we average over a short enough interval. (Geometrically: the slope of the secant line should be approximately equal to the slope of the tangent line, if the latter slope exists and if the two points on the secant line are close enough to each other.) In other words, we should expect that, if $f'(a)$ exists, then

$$\frac{f(a + \Delta x) - f(a)}{\Delta x} \approx f'(a) \quad \text{for } \Delta x \text{ small enough.} \tag{2.2.2}$$

If we multiply both sides of equation (2.2.2) by Δx, and then add $f(a)$ to both sides, we get

$$f(a + \Delta x) \approx f(a) + f'(a)\Delta x \quad \text{for } \Delta x \text{ small enough.} \tag{2.2.3}$$

We'll return to this equation *verbatim* in the next section. But for now, we want to change the notation in equation (2.2.3) just a bit, to complete the argument we are making here.

Specifically: let's give a new name to $a + \Delta x$. Let's simply call it x. Then of course $\Delta x = x - a$, so equation (2.2.3) says

$$f(x) \approx f(a) + f'(a)(x - a) \quad \text{for } x - a \text{ small enough.} \tag{2.2.4}$$

The right-hand side of (2.2.4) is, as we saw in the previous section, just the tangent line to $y = f(x)$ at $x = a$. (See equation (2.1.6).) So (2.2.4) tells us that $f(x)$ *and the tangent line to the graph of*

$f(x)$ at $x = a$ are approximately the same, if x is close to a. This is exactly what we wanted to demonstrate. And it justifies the term "locally linear:" if $f'(a)$ exists then, at least locally (near $x = a$), the graph of $f(x)$ looks like a line. (And not just any line; it looks like its tangent line at this point.)

For example, here are graphs of $f(x) = x^2$ at successively finer scales, all zeroing in on the point $x = 3$. (Compare with Examples 2.1.2 and 2.1.4 above.) Note that the axes do not intersect at (0,0) in any of these graphs except the first.

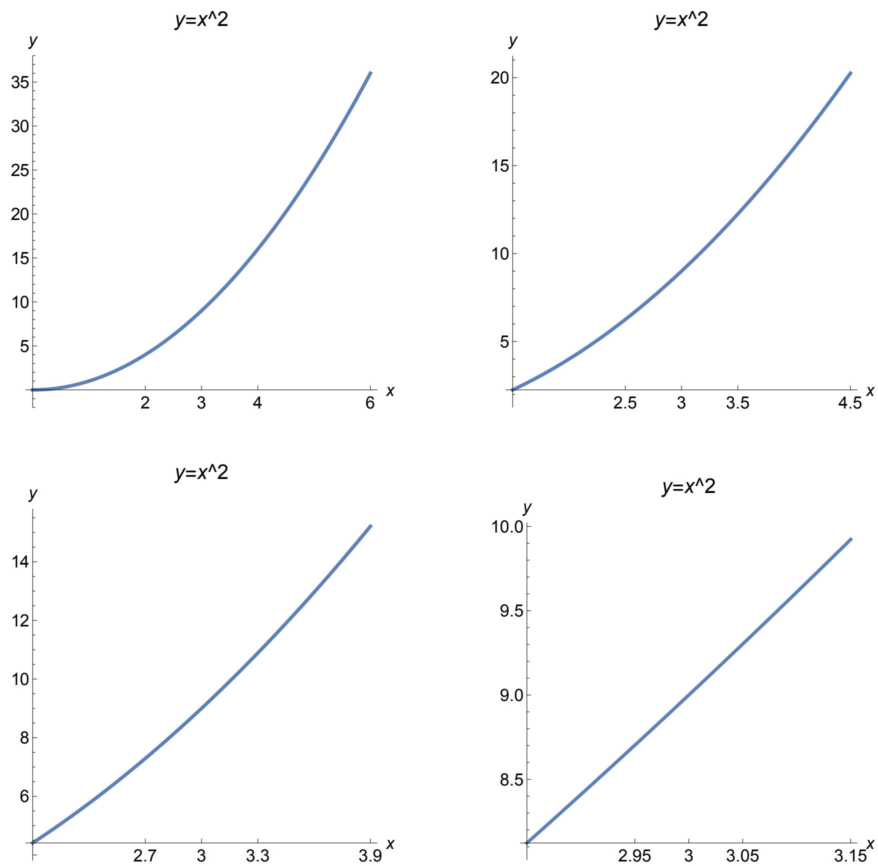


Figure 2.5. Zooming in on $f(x) = x^2$ near $x = 3$

Remark 2.2.1. The existence of $f'(a)$ does not tell us how *far* we need to zoom in, before $f(x)$ "looks linear" near $x = a$. It merely tells us that it will do so at *some* "magnification." This is to be contrasted with functions that fail to be differentiable at a given point; such functions will not look linear no matter how far we zoom in on that point. See the exercises below.

The intuition behind all of the above examples is that a function is differentiable at a point if it has no jumps, breaks, vertical tangent lines, or sharp corners at that point. For example, each of the functions of Example 2.2.1 above is differentiable at every number x *except* $x = 0$. It may be shown that polynomials are differentiable everywhere, as are the functions $y = \cos(x)$ and $y = \sin(x)$. And other trigonometric functions, such as $y = \tan(x)$, are differentiable except at points where they are undefined. (For example, $y = \cot(x)$ is nondifferentiable precisely at the integer multiples of π, since $\cot(x) = \cos(x)/\sin(x)$, and $\sin(x)$ is zero precisely when $x = k\pi$ for some integer k.) Similarly, *rational functions*, meaning functions that are defined as quotients of polynomials, e.g.

$$q(x) = \frac{x^4+1}{x^2-2x-3} \quad \text{or} \quad m(z) = \frac{45z^{46}-26z^{15}+3z-4}{14z^{100}-100z^{14}+1401},$$

are differentiable everywhere except at points where the denominator equals zero.

Continuous functions

We say that a function f is **continuous at a point** $x = a$ if

- it is defined at the point, and
- we can achieve changes in the output that are arbitrarily small by restricting changes in the input to be sufficiently small.

This second condition can also be expressed in the following form (due, in essence, to Augustin Cauchy in the early 1800's):

> Given any positive number ϵ (the proposed limit on the change in the output is traditionally designated by the Greek letter ϵ, pronounced 'epsilon'), there is always a positive number δ (the Greek letter 'delta'), such that whenever the change in the input is less than δ, then the corresponding change in the output will be less than ϵ.

A function is said to be continuous on a set of real numbers if it is continuous at each point of the set.

Cauchy's definition of continuity is quite technical. For our purposes, it will suffice to think of continuity in a more intuitive way: a function is continuous on an interval if the graph of the function has no gaps or jumps in it, on that interval. That is, a continuous function is one that you can draw without picking up your pencil.

It's not hard to see that differentiability is *stronger* than continuity. That is, if f is differentiable at $x = a$, then it must be continuous there. The converse is false: continuity need not imply differentiability. For instance, the functions f, g, and h of Example 2.2.1 are all continuous at $x = 0$, but none are differentiable there. (The function k of Example 2.2.1 is neither continuous nor differentiable at $x = 0$.)

There are examples of functions that are continuous but nondifferentiable at every single real number x! But we won't encounter such functions in this text.

Exercises

1. Using computer software (like Sage), plot the function $f(x) = 3x^5 - 5x^3$ on each of the following domains (all of which are centered on $x = 2$): [0,4], [1,3], [1.9,2.1], [1.99,2.01].

(a) Describe what you see (in terms of the apparent shape of $f(x)$, as you zoom in further and further).

(b) What one-word, or two-word, adjective from Section 2.2 above describes this function $f(x)$ at $x = 2$?

(c) From your plot over the interval [1.99,2.01], in part (a) above, choose two distinct points on the graph of $f(x)$. Label these points with their coordinates. (Read the coordinates off of the axes as well as you can; use a straightedge if it helps.)

(d) Compute and write down the slope of the line between these two points. Use this result to estimate $f'(2)$.

2. Repeat Exercise 1 above for the function $f(x) = 8\sqrt{x}$. Use the same four intervals here as you did in Exercise 1. Again, use your result to estimate $f'(2)$.

3. Repeat Exercise 1 above for the function $f(x) = 32/\sqrt{x}$. Use the same four intervals here as you did in Exercise 1. Again, use your result to estimate $f'(2)$.

4. Using computer software, plot the function $g(x) = x^2 - 10|x - 2|$ on each of the domains [0,4], [1.9,2.1], [1.99,2.01], and [1.999,2.001].

(a) Describe what you see (in terms of the apparent shape of $g(x)$, as you zoom in further and further).

(b) Does the graph of $g(x)$ look more and more like that of a line, as you zoom in more and more? Is $g(x)$ locally linear at $x = 2$, or not?

5. Repeat Exercise 4 above for the function $g(x) = x^2 - 20(x - 2)^{4/5}$. (Note: if you get an error message when trying to plot this function, you can try inputting $x^2 - 20((x - 2)^4)^{1/5}$, instead of $x^2 - 20(x - 2)^{4/5}$. This will avoid the issue of raising negative numbers to fractional exponents.)

6. At how many points, and at which points, on the interval [0,3] does the function $q(x) = 1 - 2|\sin(2\pi x)|$ fail to be locally linear? Answer by using a computer to sketch this function.

7. Note: answers to the following exercise will vary widely. Use your intuition; you don't need to know any formulas from physics.

Imagine dropping a ball from a height of four feet.

(a) Let $H(t)$ be the height of this ball at time t, with H in feet above the ground, and t in seconds from the instant the ball is dropped. Draw, by hand, a *rough sketch* of $H(t)$. Sketch over a large enough domain to capture several bounces of the ball.

Don't worry about being "to scale." (For example, you don't need specific equations that would tell you how long it would take for the ball to reach the ground.) Just try to sketch the general shape of the graph.

(b) If you were to zoom in on your graph from part (a), near the instant where the ball first hits the ground, what do you think you'd see? Draw a very rough sketch of this.

(c) Is the function that you sketched in part (a) locally linear at all points in your chosen domain? If not, at which points does it fail to be locally linear?