1.4 Functions and graphs

A number of important mathematical ideas have already emerged in our study of an epidemic. In this section we pause to consider them, because they have a "universal" character. Our aim is to get a fuller understanding of what we have done so far, so we can use the ideas in other contexts.

One of these crucial ideas, which is central to mathematics, is that of a *function.* This idea is worth highlighting:

A function is a rule that specifies how the value of one variable, the input, determines the value of a second variable, the output.

Definition of a function

That is, a function describes how one quantity depends on another. For example, in our study of a measles epidemic, the relation between the number of susceptibles S and the time t is a function. We write $S(t)$ to denote that S is a function of t. Here, the variable t is called the **input**, and the variable S is called the **output**. We think of S as *depending on* t, so t is also called the *independent variable* and S the *dependent variable*.

We can also write $I(t)$ and $R(t)$, because I and R are functions of t, too. We can even write $S'(t)$ to indicate that the rate S' at which S changes over time is a function of t.

In speaking, we express $S(t)$ as "S of t" and $S'(t)$ as "S prime of t."

Notice we say that a function is a *rule*, and not a *formula*. This is deliberate. We want the study of functions to be as broad as possible, to include various ways in which one quantity can be related to another.

So far, we have followed the standard practice in science of letting the single letter S designate both the *function* – that is, the *rule* – and the *output* of that function – that is, the *dependent variable*. Sometimes, though, we will want to make the distinction. In that case we will use two different symbols. For instance, we might write $S = f(t)$; here, we are still using S to denote the output, but the new symbol f stands for the function rule. Or we might write $y = S(t)$, in which case we are still using S to denote the rule, but the new symbol y stands for the output.

Example 1.4.1. Some other examples of functions are as follows:

1. The amount of postage you pay for a letter is a function of the weight of the letter.
2. The time of sunrise is a function of what day of the year it is.
3. The position of a car's gasoline gauge (measured in centimeters from the left edge of the gauge) is a function of the amount of gasoline in the fuel tank.
4. The volume of a cubical box is a function of the length of a side. The last is a rather special kind of function because it *can* be described by an algebraic formula: if V is the volume of the box and s is the length of a side, then $V(s) = s^3$.

5. The formula $y = x^2$ defines y as a function of x. We have not given an explicit name to this function, but of course we could: we might call the rule f, in which case we could write $f(x) = x^2$ or, to be even more complete, $y = f(x) = x^2$. Or we might avoid introducing a new letter, like f, and simply use the same letter y to denote the both the output of the function and the function itself. That is, we might sometimes write $y(x) = x^2$.

   Similarly, the formulas $y = \sqrt{x-1}$, $y = 1/\sqrt{3-2x}$, and $y = 3x - 5$ all express an output y as a function of an input x. The last of these formulas gives an example of a *linear* function. Linear functions will be discussed in detail in the next section.

6. Temperature F, in degrees Fahrenheit, is a function of temperature C, in degrees Celsius, according to the formula
$$F = \frac{9}{5}C + 32$$
   (which also describes a linear function).

7. The formula
$$P(t) = \frac{100}{1 + 9e^{-t/10}}$$
   might express population $P(t)$, in thousands, as a function of time t, in days (from a given starting point), in a certain "logistic growth" situation. We'll discuss logistic growth in more detail later.

8. A **constant function** is one that gives the same output for every input. If h is the constant function that always gives back 17, then in formula form we would express this as $h(x) = 17$. Here, it doesn't matter what the input x is. For example, $h(0) = 17$, $h(-35) = 17$, $h(47\pi) = 17$, $h(\text{whatever}) = 17$!

9. Water density D, in kilograms per cubic meter (kg/m^3), is a function of water temperature C, in degrees Celsius; we might write $D = q(C)$.

Some technical details

Domain and range. The set of values that the input to a function takes is called the **domain** of the function. The domain may depend on the contexts, both physical and mathematical. If no physical context is given, then the domain is sometimes called the **natural domain**. This terminology is perhaps a bit misleading, in that the natural domain is the domain that applies in the *absence* of any natural, "real-world" constraints. But it is what it is.

For example, the function defined by $y = 1/\sqrt{3-2x}$ has natural domain equal to
$$\{\text{real numbers } x \colon x < 3/2\} \tag{1.4.1}$$
(the set of all real numbers x that are less than $3/2$). Why? Because, mathematically, one can neither divide by zero nor take the square root of a negative number. So in the formula

$y = 1/\sqrt{3-2x}$, we can neither have $3 - 2x = 0$ or $3 - 2x < 0$. So we must have $3 - 2x > 0$, or $3 > 2x$, or $3/2 > x$, or $x < 3/2$. In interval notation, the set of such x may be denoted $(-\infty, 3/2)$.

Next, consider the formula $F = \frac{9}{5}C + 32$ in item 6 of the above example. All by itself, this formula defines a function with natural domain equal to the set of all real numbers, denoted $(-\infty,\infty)$ or, sometimes, $\mathbb{R}$. This is because $\frac{9}{5}C + 32$ makes mathematical sense for any real number C.

But in this case, the formula is not the complete picture. A physical context for the formula $F = \frac{9}{5}C + 32$ was explicitly stated, which puts restrictions on our domain. Namely, since $-273°$ Celsius is absolute zero, a reasonable domain to ascribe to this situation is

$$(-273, \infty) \quad \text{or} \quad \{C \in \mathbb{R} \colon C > -273\}.$$

One might argue that absolute zero is theoretically attainable, in which case one might take the domain to be $[-273, \infty)$. At the other end, contemporary models postulate a maximum attainable "Planck temperature" T_P equal to about 1.417×10^{32} degrees Celsius, so maybe the domain here should be $(-273, 1.417 \times 10^{32})$. (Or $[-273, 1.417 \times 10^{32}]$?)

The moral is that domains can sometimes be open to interpretation! (This is true even of *natural* domains. For example, if one allows for *complex numbers*, then one *can* take the square root of a negative number. In this text, though, unless otherwise stated, we will allow only for real numbers.)

The set of values taken by the output of a function is called the **range** of the function. This will depend on the domain. For example, if we take $[-273, 1.417 \times 10^{32}]$ as the domain of a function given by the formula $F = \frac{9}{5}C + 32$, then the range of this function is

$$\left[\frac{9}{5}(-273) + 32,\ \frac{9}{5}(1.417 \times 10^{32}) + 32\right] = [-459.4, 2.5506 \times 10^{32}].$$

Input versus output. Note the use of the words "rule," "specifies," and "determines" in our definition of function, above. These words all highlight an *essential* property of any function: a function associates a *unique* output to each particular input. Another word for "unique," in this context, is "unambiguous."

For example, the formula $y = x^2$ defines y as a function of x, because given x, we know exactly what y is: it's the square of x. If $x = 3$, we know unambiguously that $y = 3^2 = 9$, and so on.

As a consequence of our definition of function, the statement "y is a function of x" **need not** imply that x is a function of y. Indeed, the formula $y = x^2$ does not give x as a function of y. If we choose $y = 16$, for example, there is ambiguity as to what x must be: x *could* equal 4, but it could also equal -4, since both of these numbers satisfy the equation $16 = x^2$.

Of course, many functions *do* specify input and output uniquely in terms of each other. Such functions are sometimes said to be *one-to-one*. For example, the equation $F = \frac{9}{5}C + 32$ describes a one-to-one function; this function does give C uniquely in terms of F (in addition to giving F uniquely in terms of C). Specifically, we can solve this equation for C to get the unambiguous formula

$$C = \frac{5}{9}(F - 32).$$

We'll return to the topic of one-to-one functions in a later chapter.

Function notation; chaining, or composing, functions

It is important not to confuse an expression like $S(t)$ with a product; $S(t)$ does *not* mean $S \times t$. On the contrary, the expression $S(1.4)$, for example, stands for the output of the function S when 1.4 is the input. In the epidemic model, we interpret this as the number of susceptibles that remain 1.4 days after today (or whatever day we designate as $t = 0$).

The symbols we use to denote the input and the output of a function are just names; if we change them, we don't change the function. For example, here are four ways to describe the same function g:

$$\begin{aligned} g &: \text{multiply the input by 5, then subtract 3;} \\ g(x) &= 5x - 3; \\ g(u) &= 5u - 3; \\ g(\text{whatever}) &= 5 \times \text{whatever} - 3. \end{aligned}$$

.

It is important to realize that the *formulas* we just wrote in the last three lines are merely shorthand for the instructions stated in the first line.

If you keep this in mind, then complex-looking combinations like $g(g(2))$ can be decoded easily by remembering g of *anything* is just 5 times that anything, minus 3. We could thus evaluate $g(g(2))$ from the inside out:

$$g(g(2)) = g(5 \cdot 2 - 3) = g(7) = 5 \cdot 7 - 3 = 32,$$

or we could evaluate it from the outside in:

$$g(g(2)) = 5g(2) - 3 = 5(5 \cdot 2 - 3) - 3 = 5 \cdot 7 - 3 = 32,$$

as before. It is often, though by no means always, easier to evaluate combinations like $g(g(2))$ from the inside out.

Suppose f is some other rule, say $f(x) = x^2 - 1$. Remember that this is just shorthand for "Take the input (whatever it is), square it, and subtract 1," or "f of whatever is whatever squared, minus one." We could then evaluate

$$f(g(3)) = f(5 \cdot 3 - 3) = f(12) = 12^2 - 1 = 144 - 1 = 143,$$

while

$$g(f(3)) = g(3^2 - 1) = g(8) = 5 \times 8 - 3 = 37.$$

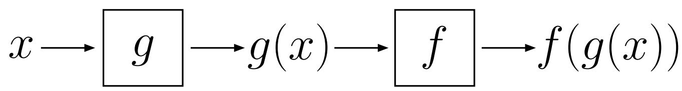
More generally, we have

$$f(g(x)) = f(5x - 3) = (5x - 3)^2 - 1 = 25x^2 - 30x + 9 - 1 = 25x^2 - 30x + 8$$

and

$$g(f(x)) = g(x^2 - 1) = 5(x^2 - 1) - 3 = 5x^2 - 5 - 3 = 5x^2 - 8.$$

When we use the output of one function as the input for another, we say that we are *chaining*, or *composing*, these functions. More specifically, the function $y = f(g(x))$ obtained by taking the output of g, and using this output as the input to f, is called *the composition* of f and g (denoted $f \circ g$ in some references, though we will not use this notation). Schematically, the picture is this:



The composition, or chain, of f and g

Warning: the composition of f and g is not, in general, the same as the composition of g and f; our above example illustrates a situation where $f(g(x)) \neq g(f(x))$.

To avoid possible confusion, we will usually say things like "the composition $f(g(x))$" rather than "the composition of f and g," since the latter terminology does not emphasize the order is which the composition is done.

Note that chaining is not limited to situations involving only two functions.We can chain together any number of functions: For example, if f and g are as above, and $h(x) = 3 - x$, then we can form the function $y = g(h(f(x)))$, defined as follows (working from the inside out):

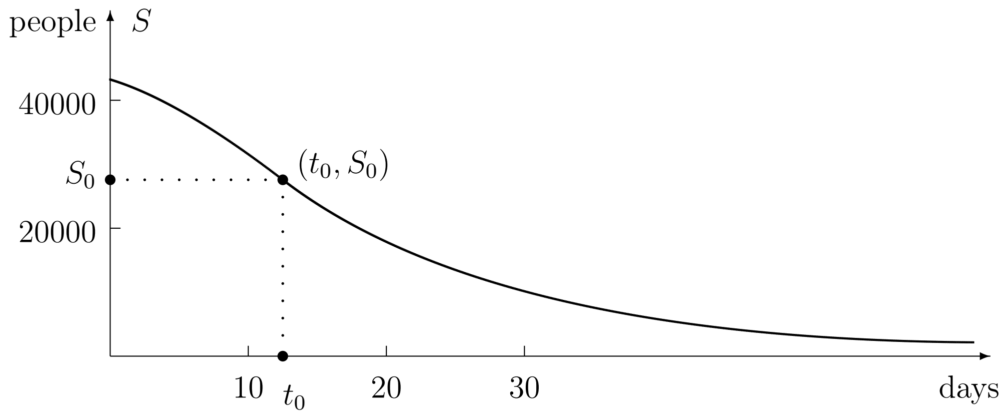
$$\begin{aligned} g(h(f(x))) &= g(h(x^2 - 1)) = g(3 - (x^2 - 1)) = g(3 - x^2 + 1) \\ &= g(4 - x^2) = 5(4 - x^2) - 3 = 20 - 5x^2 - 3 = 17 - 5x^2. \end{aligned}$$

And so on.

Chaining will turn out to be very important later in this course. For now, though, you should treat it simply as part of the formal language of mathematics. It is somewhat analogous to learning how to conjugate verbs in French class – it's perhaps not very exciting for its own sake, but it allows us to read the interesting stuff later on.

Graphs

A graph describes a function in a visual form. Sometimes – as with a seismograph or a lie detector, for instance – this is the *only* description we have of a particular function. The usual arrangement is to put the input variable on the horizontal axis and the output on the vertical – but it is a good idea when you are looking at a particular graph to take a moment to check; sometimes, the opposite convention is used! This is often the case in geology and economics, for instance.



Sketched above is the graph of a function $S(t)$ that tells how many susceptibles there are after t days. Given any t_0, we "read" the graph to find $S(t_0)$, as follows: from the point t_0 on the t-axis, go vertically until you reach the graph; then go horizontally until you reach the S-axis. The value S_0 at that point is the output $S(t_0)$. Here t_0 is about 13 and S_0 is about 27,000; thus, the graph says that $S(13) \approx 27{,}000$, or about 27,000 susceptibles are left after 13 days.

The circular functions

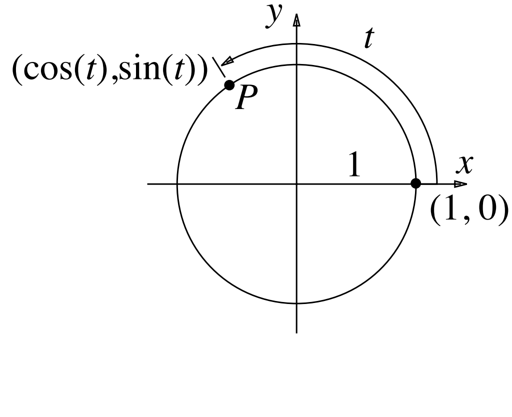
Graphing packages "know" the familiar functions of trigonometry. Trigonometric functions are qualitatively different from the functions in the preceding problems. Those functions are defined by algebraic formulas (that is, formulas involving only addition, subtraction, multiplication, division, exponentiation, and roots), so they are called **algebraic functions**. The trigonometric functions are defined by explicit "recipes," but *not* by algebraic formulas; they are called **transcendental functions**. For calculus, we usually use the definition of the trigonometric functions as **circular functions**. This definition begins with a unit circle centered at the origin. Given the input number t, locate a point P on the circle by tracing an arc of length t along the circle from the point (1,0). If t is positive, trace the arc counterclockwise; if t is negative, trace it clockwise. Because the circle has radius 1, the arc of length t subtends a central angle of **radian** measure t.

The circular (or trigonometric) functions $\cos t$ and $\sin t$ are defined as the coordinates of the point P,

$$P = (\cos(t), \sin(t)).$$

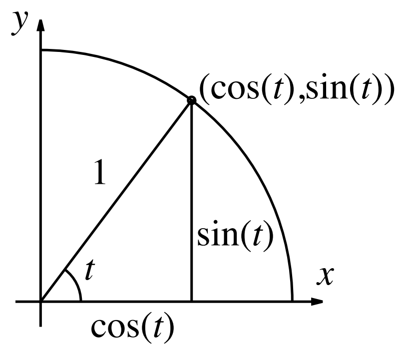
The other trigonometric functions are defined in terms of the sine and cosine:

$$\tan(t) = \sin(t)/\cos(t), \qquad \sec(t) = 1/\cos(t),$$
$$\cot(t) = \cos(t)/\sin(t), \qquad \csc(t) = 1/\sin(t).$$



Notice that when t is a positive acute angle, the circle definition agrees with the right triangle definitions of the sine and cosine:

$$\sin(t) = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \text{and} \qquad \cos(t) = \frac{\text{adjacent}}{\text{hypotenuse}}.$$



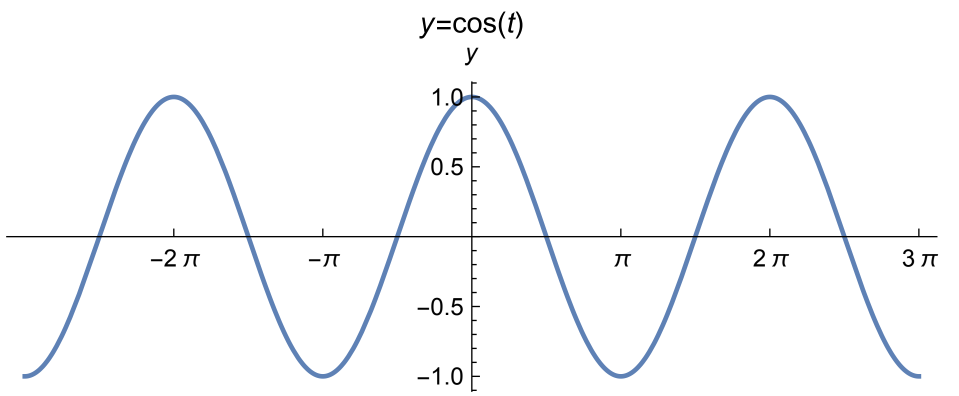
However, the circle definitions of the sine and cosine have the important advantage that they produce functions whose domains are the set of *all* real numbers. (What are the domains of the tangent, secant, cotangent and cosecant functions?)

In calculus, angles are also always measured in radians. To convert between radians and degrees, notice that the circumference of a unit circle is 2π, so the radian measure of a semi-circular arc is half of this, and thus we have

$$\pi \text{ radians } = 180 \text{ degrees.}$$

As the course progresses, you will see why radians are used rather than degrees (or mils or any other unit for measuring angles) – it turns out that the formulas important to calculus take their simplest form when angles are expressed in radians.

If one graphs the cosine of an angle on the vertical axis, against the radian measure of that angle on the horizontal axis, then, one gets a picture like this:



The graph of $y = \sin(t)$ looks similar, but shifted to the right by $\pi/2$ radians (so that the graph of the sine function goes though the origin (0,0)).

Computer graphing packages "know" the trigonometric functions in radian form. You might wonder, though, how a computer or calculator "knows" that $\sin(1) = .017452406\ldots$. It certainly isn't drawing a very accurate circle somewhere and measuring the y coordinate of some point. While the circular function approach is a useful way to think about the trigonometric functions conceptually, it isn't very helpful if we actually want values of the functions. One of the achievements of calculus, as you will see later in this course, is that it provides effective methods for computing values of functions like the circular functions that aren't given by algebraic formulas.

Functions of Several Variables

Language and notation. Many functions depend on more than one variable. For example, sunrise depends on the day of the year but it also depends on the latitude (position north or south of the equator) of the observer. Likewise, the crop yield from an acre of land depends on the amount of fertilizer used, but it also depends on the amount of rainfall, on the composition of the soil, on the amount of weeding done – to mention just a few of the other variables that a farmer has to contend with.

The rate equations in the SIR model also provide examples of functions with more than one input variable. The equation

$$I' = .00001\,SI - I/14$$

says that we need to specify both S and I to find I'. We can say that

$$F(S, I) = .00001\,SI - I/14$$

is a function whose input is the **ordered pair** of variables (S, I). In this case F is given by an algebraic formula. While many other functions of several variables also have formulas – and they are extremely useful – not all functions do. The sunrise function, for example, might be given by a table that shows the time of sunrise for different days of the year and different latitudes.

As a technical matter it is important to note that the input variables S and I of the function $F(S, I)$ above appear in a particular *order*, and that order is part of the definition of the function. For example, $F(1, 0) = 0$, but $F(0, 1) = -1/14$. (Do you see why? Work out the calculations yourself.)

Parameters. Suppose we rewrite the rate equation for I', replacing .00001 and $1/14$ with the general values a and b:

$$I' = aSI - bI.$$

This makes it clear that I' depends on a and b, too. But note that a and b are not variables in quite the same way that S and I are. For example, a and b will vary if we switch from one disease to another or from one population to another. However, they will stay fixed while we consider a particular disease in a particular population. By contrast, S and I will *always* be treated as variables. We call a quantity like a or b a **parameter**.

To emphasize that I' depends on the parameters as well as S and I, we can write I' as the output of a new function

$$I' = I'(S, I, a, b) = aSI - bI$$

whose input is the set of *four* variables (S, I, a, b), in that order. The variables S, I, and R must also depend on the parameters, too, and not just on t. Thus, we should write $S(t, a, b)$, for example, instead of simply $S(t)$. We implicitly used the fact that S, I, and R depend on a and b when we discovered there was a threshold value S_T for an epidemic.

There are even more parameters lurking in the SIR problem. To uncover them, recall that we needed *two* pieces of information to estimate S, I, and R over time:

(1) the rate equations;

(2) the initial values $S(0)$, $I(0)$, and $R(0)$.

We used $S(0) = 45{,}400$, $I(0) = 2{,}100$, and $R(0) = 2{,}500$ in the text, but if we had started with other values then S, I, and R would have ended up being different functions of t. Thus, we should really write

$$S = S(t, a, b, S(0), I(0), R(0))$$

to tell a more complete story about the inputs that determine the output S. Most of the time, though, we do *not* want to draw attention to the parameters; we usually write just $S(t)$.

In our study of the SIR model, it was natural not to separate functions that have one input variable from those that have several. This is the pattern we shall follow in the rest of the course. In particular, we will want to deal with parameters, and we will want to understand how the quantities we are studying depend on those parameters.

Exercises

Part 1: Functions: evaluation, composition, domain (Exercises 1–4)

The next four exercises refer to these functions:

$c(x,y) = 17$	a constant function
$j(z) = z$	the identity function
$r(u) = 1/u$	the reciprocal function
$D(p,q) = p - q$	the difference function
$s(y) = y^2$	the squaring function
$\ell(x) = 3 - x$	a linear function
$Q(v) = \dfrac{2v+1}{3v-6}$	a rational function
$H(x) = \begin{cases} 5 & \text{if } x < 0 \\ x^2 + 2 & \text{if } 0 \le x < 6 \\ 29 - x & \text{if } 6 \le x \end{cases}$	a "piecewise" function
$T(x,y) = r(x) + Q(y)$	

1. Determine the following values:

$c(5,-3)$	$s(17)$	$c(a,b)$	$j(u^2+1)$
$j(c(3,-5))$	$\ell(1.1)$	$r(1/17)$	$Q(0)$
$Q(2)$	$Q(3/7)$	$D(5,-3)$	$D(-3,5)$
$H(1)$	$H(7)$	$\ell(4)$	$H(H(H(-3)))$
$r(Q(3))$	$Q(r(3))$	$T(3,7)$	$T(s(2),j(3))$
$r(s(-4))$	$r(r(r(r(r(u)))))$	$\ell(\text{whatever})$	$D(\text{mellow}, \text{yellow})$

2. True or false. Give reasons for your answers: if you say true, explain why; if you say false, give an example that shows why it is false.

(a) For every non-zero number x, $r(r(x)) = j(x)$.

(b) $c(\pi, -965.32) = j(17)$.

(c) If $a > 1$, then $s(a) > 1$.

(d) If $a > b$, then $s(a) > s(b)$.

(e) For all real numbers a and b, $s(a + b) = s(a) + s(b)$.

(f) For all real numbers x, $s(r(x)) = r(s(x))$.

(g) For all real numbers x, $s(\ell(x)) = \ell(s(x))$.

(h) For all real numbers a, b, and c, $D(D(a,b),c)) = D(a,D(b,c))$.

3. Find all numbers x for which $Q(x) = r(Q(x))$.

4. Recall that the **natural domain** of a function f is the largest possible set of real numbers x for which $f(x)$ is defined. For example, the natural domain of $r(x) = 1/x$ is the set of all *non-zero* real numbers.

(a) Find the natural domains of Q and H.

(b) Find the natural domains of $P(z) = Q(r(z))$; $R(v) = r(Q(v))$.

(c) What is the natural domain of the function $W(t) = \sqrt{\dfrac{1-t^2}{t^2-4}}$?

Part 2: Graphing functions using software (like Sage) (Exercises 5–13)

Exercises 5–13 are intended to give you some experience using a "graphing package" on a computer. This is a program that will draw the graph of a function $y = f(x)$ whose formula you know. You must type in the formula, using the following symbols to represent the basic arithmetic operations:

to indicate	type
addition	`+`
subtraction	`-`
multiplication	`*`
division	`/`
an exponent	`^`

The caret " `^` " appears above the "`6`" on a keyboard (Shift-6).

Here is an example:

to enter:	type:
$\dfrac{7x^5 - 9x^2}{x^3 + 1}$	`(7*x^5 - 9*x^2)/(x^3 + 1)`

The parentheses you see here are important. If you do not include them, the computer will interpret your entry as

$$7x^5 - \frac{9x^2}{x^3} + 1 = 7x^5 - \frac{9}{x} + 1 \neq \frac{7x^5 - 9x^2}{x^3 + 1}.$$

In some graphing packages, you do not need to use `*` to indicate a multiplication. If this is true for the package you use, then you can enter the fractional expression above in the somewhat simpler form

```
(7x^5 - 9x^2)/(x^3 + 1).
```

To do the following exercises, follow the specific instructions for the graphing package you are using. (In Sage, for example, you *do* need to use `*` to indicate multiplication. Sage will not understand `7x`; you would need to type in `7*x`.)

5. Graph the following functions. Put labels and scales on the axes.

(a) $y = 3x - 1$;

b) $y = 600 - x^3$.

6. Sketch the graph of each of the following functions. Put labels and scales on the axes. For each graph that you draw, indicate (i) its y-intercept; and (ii) its x-intercept(s).

For part (d) you will need the **quadratic formula**

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

for the roots of the **quadratic equation** $ax^2 + bx + c = 0$.

(a) $y = x^2$

(b) $y = x^2 + 1$

(c) $y = (x + 1)^2$

(d) $y = 3x^2 + x - 1$

7. Graph the function $f(x) = .6x + 2$ on the interval $-4 \leq x \leq 4$.

(a) What is the y-intercept of this graph? What is the x-intercept?

(b) Read from the graph the value of $f(x)$ when $x = -1$ and when $x = 2$. What is the difference between these y values? What is the difference between the x values? According to these differences, what is the slope of the graph? According to the *formula*, what is the slope?

8. Graph the function $f(x) = 1 - 2x^2$ on the interval $-1 \leq x \leq 1$.

(a) What is the y-intercept of this graph? The graph has two x-intercepts; use algebra to find them.

You can also find an x-intercept using the computer. The idea is to **magnify** the graph near the intercept until you can determine as many decimal places in the x coordinate as you want. For a start, graph the function on the interval $0 \le x \le 1$. You should be able to see that the graph on your computer monitor crosses the x-axis somewhere around .7. Regraph $f(x)$ on the interval $.6 \le x \le .8$. You should then be able to determine that the x-intercept lies between .70 and .71. This means $x = .7\ldots$; that is, you know the location of the x-intercept to one decimal place of accuracy.

(b) Regraph $f(x)$ on the interval $.70 \le x \le .71$, to get two decimal places of accuracy in the location of the x-intercept. Continue this process until you have at least 4 places of accuracy. What is this x-intercept, to four places?

9. The following exercise lets you review the trigonometric functions and explore them using computer graphing.

(a) Graph the function $f(x) = \sin(x)$ on the interval $-2 \le x \le 10$.

(b) What are the x-intercepts of $\sin(x)$ on the interval $-2 \le x \le 10$? Determine them to two decimal places accuracy.

(c) What is the largest value of $f(x)$ on the interval $-2 \le x \le 10$? Which value of x makes $f(x)$ largest? Determine x to two decimal places accuracy.

(d) Regraph $f(x)$ on the small interval $-0.01 \le x \le 0.01$. Describe what you see. Estimate the slope of this graph at $x = 0$.

(e) Graph the function $f(x) = \cos(x)$ on the domain $0 \le x \le 14$. On the same set of axes, graph the function $g(x) = \cos(2x)$.

(f) How far apart are the x-intercepts of $f(x)$? How far apart are the x-intercepts of $g(x)$?

(g) The graph of $g(x)$ has a pattern that repeats. How wide is this pattern? The graph of $f(x)$ also has a repeating pattern; how wide is *it*?

(h) Compare the graphs of $f(x)$ and $g(x)$ to one another. In particular, can you say that one of them is a stretched or compressed version of the other? Is the compression (or stretching) in the vertical or the horizontal direction?

(i) Construct a *new* function $h(x)$ whose graph is the same shape as the graph of $g(x) = \cos(2x)$, but make the graph of $h(x)$ twice as tall as the graph of $g(x)$. [A suggestion: either deduce what $h(x)$ should be, or make a guess. Then test your choice on the computer. If your choice doesn't work, think how you might modify it, and then test your modifications the same way.] Graph h on the same set of axes as you did f and g.

10. The aim here is to find a solution to the equation $\sin x = \cos(3x)$. There is no purely *algebraic* procedure to solve this equation. Because the sine and cosine are not defined by *algebraic* formulas,

this should not be particularly surprising. (Even for algebraic equations, there are only a few very special cases for which there are formulas like the quadratic formula. In chapter 5 we will look at a method for solving equations when formulas can't help us.)

(a) Graph the two functions $f(x) = \sin(x)$ and $g(x) = \cos(3x)$ on the interval $0 \le x \le 1$.

(b) Find a solution of the equation $\sin(x) = \cos(3x)$ that is accurate to six decimal places.

(c) Find *another* solution of the equation $\sin(x) = \cos(3x)$, accurate to four decimal places. Explain how you found it.

11. Use a graphing program to make a sketch of the graph of each of the following functions. In each case, make clear the domain and the range of the function, where the graph crosses the axes, and where the function has a maximum or a minimum.

a) $F(w) = (w-1)(w-2)(w-3)$

b) $Q(a) = \dfrac{1}{a^2+5}$

c) $E(x) = x + \dfrac{1}{x}$

d) $e(x) = x - \dfrac{1}{x}$

e) $g(u) = \sqrt{\dfrac{u-1}{u+1}}$

f) $M(u) = \dfrac{u^2-2}{u^2+2}$

12. Graph, on the same set of axes, the following three functions:

$$f(x) = 2^x, \qquad g(x) = 3^x, \qquad h(x) = 10^x.$$

Use the domain $-1 \le x \le 1$.

(a) Which function has the largest value when $x = -1$?

(b) Which is climbing most rapidly when $x = 0$?

(c) Magnify the picture at $x = 0$ by resetting the size of the domain to $-0.01 \le x \le 0.01$. Describe what you see. Estimate the slopes of the three graphs at $x = 0$.