1.3 Prediction using SIR

Euler's method amounts to the following **BIG IDEA** for using "rate equations," like the above SIR equations, to predict: if Q is any quantity, varying with time, then between any two instants – a "new" one and an "old" one, say – we have

$$\text{New } Q = \text{Old } Q + \Delta Q \tag{a}$$

where ΔQ denotes the *change* in Q, from the "old" instant to the "new" instant. Moreover, we have

$$\Delta Q \approx Q' \, \Delta t \tag{b}$$

where Δt is the elapsed time, and Q' is the rate of change of Q with respect to time. (See, for example, the prediction equation (PEGV) of Section 1.1 above.) Again, equation (b) simply says: net change in a quantity equals the rate of change of that quantity, times elapsed time.

Remark 1.3.1. As noted in Section 1.1, the "≈" in equation (b) above means "is approximately equal to." Why are the two sides of equation (b) only approximately equal to each other? Because Q' itself is (typically) changing. Distance traveled, for example, only equals speed times time if the speed is constant over the interval of time in question. If the speed is changing, then reading the speedometer at a given instant will not allow us to predict the distance traveled over the next hour *exactly* – or even very well, most likely. On the other hand, it will probably give a pretty good idea of distance traveled over the next second, say.

The upshot is that equation (b) should be "pretty accurate" if Δt is small. How small, and how accurate? These questions will be explored in depth as we proceed. For now, we'll understand equation (b) in the sense – admittedly, a somewhat vague sense – just discussed.

Now, let's *use* equations (a) and (b), together with the above SIR equations, to predict, as follows.

Example 1.3.1. Consider a disease that behaves according to the above SIR model. Suppose the initial values $S(0)$, $I(0)$, and $R(0)$ of S, I, and R are given by

$$S(0) = 500, \quad I(0) = 10, \quad R(0) = 0.$$

As before, we'll take the units of S, I, and R to be persons, and the units of time t to be days. Let's suppose we also know that the transmission and recovery coefficients a and b are given by

$$a = 0.001 \text{ (person-day)}^{-1}, \quad b = 0.2 \text{ day}^{-1}.$$

Use this information to predict $S(4)$, $I(4)$, and $R(4)$, using

(i) stepsize $\Delta t = 2$;

(ii) stepsize $\Delta t = 4$.

Solution.

(i) As we're starting at $t = 0$, and using stepsize $\Delta t = 2$, the first values of S, I and R to be predicted are $S(2)$, $I(2)$, and $R(2)$. Let's begin with $S(2)$. We have

$$\begin{aligned} S(2) &= S(0) + \Delta S && \text{(by (a))} \\ &\approx S(0) + S'(0)\Delta t && \text{(by (b))} \\ &= S(0) + (-aS(0)I(0))\Delta t && \text{(by the } SIR \text{ equations)} \\ &= 500 + (-0.001 \times 500 \times 10) \times 2 && \text{(plug in numerical values)} \\ &= 500 - 10 = 490. \end{aligned}$$

Remark 1.3.2. Because an approximation occurs *somewhere* (anywhere!) in the computation of $S(2)$, the final result of that computation is itself an approximation. So, in spite of the "=" appearing in the last step (and in various other steps) of the above computation, what we have actually found is that $S(2) \approx 490$, and not that $S(2) = 490$.

Next, we compute $R(2)$ (we'll save $I(2)$ for last, because the equation for I' is less simple than the one for R'):

$$\begin{aligned} R(2) &= R(0) + \Delta R && \text{(by (a))} \\ &\approx R(0) + R'(0)\Delta t && \text{(by (b))} \\ &= R(0) + (bI(0))\Delta t && \text{(by the } SIR \text{ equations)} \\ &= 0 + (0.2 \times 10) \times 2 && \text{(plug in numerical values)} \\ &= 0 + 4 = 4. \end{aligned}$$

To find $I(2)$, we use the fact that, by assumption, $S + I + R$ is constant. Since, initially (at $t = 0$), this sum equals $500 + 10 + 0 = 510$, we have

$$I(2) = 510 - S(2) - R(2) \approx 510 - 490 - 4 = 16.$$

To summarize our first "step" of part (i) of this example: we've found that

$$S(2) \approx 490, \quad I(2) \approx 16, \quad R(2) \approx 4 \quad \text{(persons)}. \tag{1.3.1}$$

For the next step – estimating $S(4)$, $I(4)$, and $R(4)$ – we imagine now that $t = 2$ is our "old," or starting, value of t, and that $t = 4$ is our "new," or final, value of t. We then proceed as above, using the (approximate) values of $S(2)$, $I(2)$, and $R(2)$ just computed, and summarized in equations (1.3.1). So, by the same kind of reasoning as we used in the first step,

$$\begin{aligned} S(4) &= S(2) + \Delta S \\ &\approx S(2) + S'(2)\Delta t \\ &= S(2) + (-aS(2)I(2))\Delta t \\ &\approx 490 + (-0.001 \times 490 \times 16) \times 2 \\ &= 490 - 15.68 = 474.32. \end{aligned}$$

Note that, this time, we use the symbol "≈" in two different instances – the first time because, as before, net change is only approximately equal to rate of change times elapsed time; the second time because the "old" values of S, I, and R that we're using (that is, the values at $t = 2$) are, themselves, approximations. (And never mind the 32 hundredths of a person who is presumably part of this susceptible population at $t = 4$. While math may be used to model real life, the two aren't the same, which is probably a good thing for that 0.32 of a person.)

Similarly,

$$\begin{aligned} R(4) &= R(2) + \Delta R \\ &\approx R(2) + R'(2)\Delta t \\ &= R(2) + (bI(2))\Delta t \\ &\approx 4 + (0.2 \times 16) \times 2 \\ &= 4 + 6.4 = 10.4, \end{aligned}$$

and

$$I(2) = 510 - S(4) - R(4) \approx 510 - 474.32 - 10.4 = 25.28.$$

In sum, then: using $\Delta t = 2$, we have found that

$$S(4) \approx 474.32, \quad I(4) \approx 25.28, \quad R(4) \approx 10.4 \quad \text{(persons)}. \tag{1.3.2}$$

(ii) In much the same manner as above, we find that

$$\begin{aligned} S(4) &= S(0) + \Delta S \\ &\approx S(0) + S'(0)\Delta t \\ &= S(0) + (-aS(0)I(0)) \times \Delta t \\ &= 500 + (-0.001 \times 500 \times 10) \times 4 \\ &= 500 - 20 = 480, \end{aligned} \qquad \begin{aligned} R(4) &= R(0) + \Delta R \\ &\approx R(0) + R'(0)\Delta t \\ &= R(0) + (bI(0))\Delta t \\ &= 0 + (0.2 \times 10) \times 4 \\ &= 0 + 8 = 8, \end{aligned}$$

and

$$I(4) = (S(0) + I(0) + R(0)) - S(4) - R(4) \approx (500 + 10 + 0) - 480 - 8 = 22.$$

In sum: using $\Delta t = 4$, we find that

$$S(4) \approx 480, \quad I(4) \approx 22, \quad R(4) \approx 8 \quad \text{(persons)}. \tag{1.3.3}$$

Compare (1.3.3) with (1.3.2): not surprisingly, these estimates are different. Again, the smaller stepsize $\Delta t = 2$ yields better results than $\Delta t = 4$, because the rates of change S, I, and R are themselves continuously changing. And the smaller Δt is, the more often we recalibrate, to adjust for this change.

Using essentially the "Euler's method" algorithm implemented above, but with stepsize $\Delta t = 0.001$, we would find that

$$S(4) \approx 463.57, \quad I(4) \approx 31.30, \quad R(4) \approx 15.13 \quad \text{(persons)}.$$

These numbers are still approximations, but they are closer to the truth.

Note that, to approximate $S(4)$, $I(4)$, and $R(4)$ using stepsize $\Delta t = 0.001 = 1/1000$, we need to compute $S(t)$, $I(t)$, and $R(t)$ at $4 \times 1{,}000 = 4{,}000$ different values of t (not including the initial point $t = 0$)![1] Needless to say, we would not, and did not, do these computations by hand. We used a computer, together with the open-source "Sage" mathematical software package, which is very similar to Matlab, Mathematica, and other mathematical software that you may have seen.

See the exercises below for a Sage computer implementation of Euler's method to the SIR model.

If we could somehow make sense of the above algorithm for the case $\Delta t = 0$, we would, in theory, have an *exact* solution to the SIR equations. As it turns out, the SIR system of rate equations does *not*, in fact, admit an exact, "closed-form" solution, meaning one where $S(t)$, $I(t)$, and $R(t)$ can be written as mathematical expressions in the variable t. Still, many other interesting "real-life" phenomena do. We'll discuss all of this further in the course of this text.

Summary: Euler's method and SIR

Schematically, Euler's method, as applied to the SIR system of rate equations, looks like this:

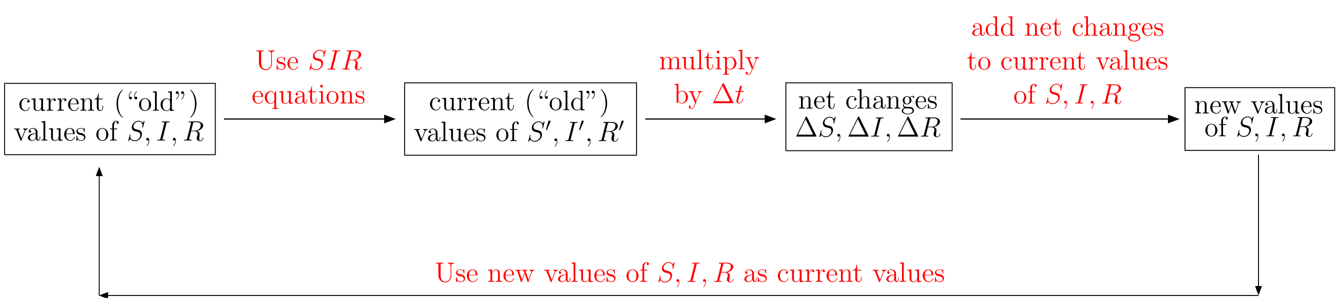


Figure 1.1. Flowchart for Euler's method applied to the SIR equations

The above flowchart is by no means complete. In particular, it doesn't indicate how, and where, one starts or ends the indicated "loop."

The starting point of this loop corresponds to using $S(0)$, $I(0)$, and $R(0)$ in the upper left-hand box of the flowchart. And what about the ending point? If one wants to predict all the way

[1] If we include the point $t = 0$, then we have 4,001 different values of t at which we are computing. In general, if we have a phenomenon that spans n intervals of time – $n = 4{,}000$ in the discussion above – and we want to observe this phenomenon at the beginning and the end of each interval, then we need $n + 1$ observations. This situation is often called the "fencepost phenomenon" – if you have 4,000 lengths of fence, then you'll need 4,001 fenceposts to support them; if you have n lengths of fence, you'll need $n + 1$ fenceposts to support them.

out to time T, say, and if one chooses a stepsize Δt, then one will need cycle through the above loop $T/\Delta t$ times. (Including the initial values $S(0)$, $I(0)$, and $R(0)$, one will then end up with $(T/\Delta t) + 1$ different values of S, I, and R. See the footnote just above.)

Threshold value of S

We conclude this section with a nice consequence of the above SIR model – one that does *not* require Euler's method or any similar iterative process. Namely: we use the above SIR equations to determine the so-called *threshold value* of the variable S.

Here's the idea. As noted earlier, the susceptible subpopulation only decreases in size. Eventually, this subpopulation will become small enough that it can no longer sustain growth in the infected population (assuming the latter subpopulation is, initially, growing). At this point, I will *peak*, and thereafter will dwindle.

The question that we wish to address is: how small is small enough? How small *does* the susceptible population need to become before I peaks, and begins to decline? We'll answer in a moment, but first, let's give a *name* to this particularly important value of S.

Definition 1.3.1. In the above SIR model, the *threshold value of* S, denoted S_T, is the value of S at which I peaks.

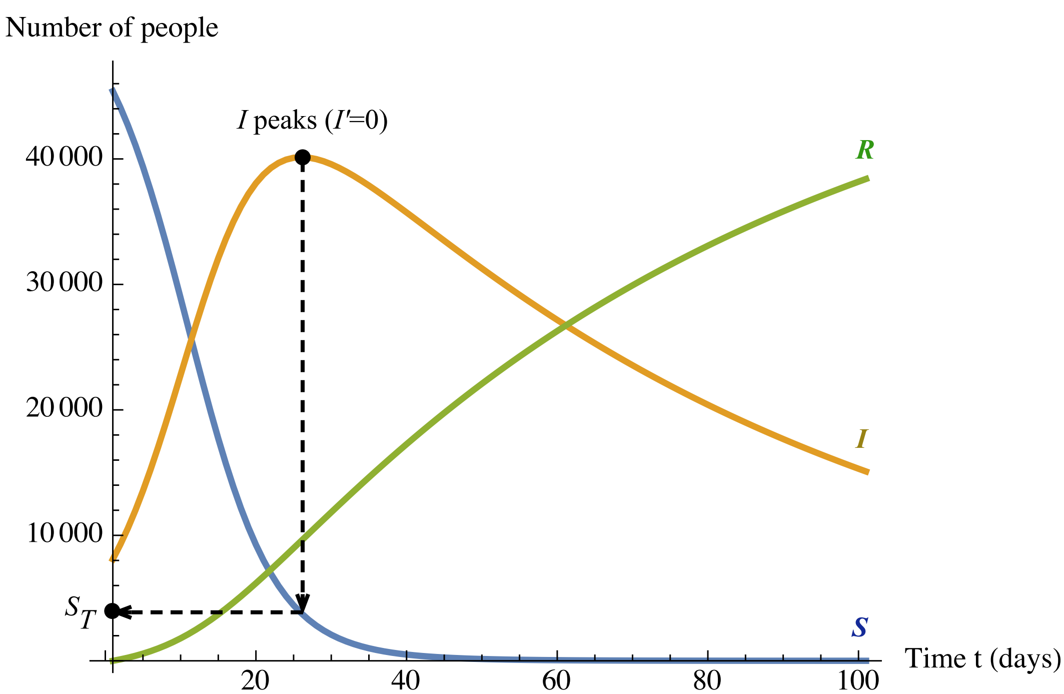


Figure 1.2. Meaning of the threshold value of S

Figure 1.2 gives a graphical interpretation of S_T. We can also use our above SIR equations – in particular, the equation for I' – to deduce a *formula* for S_T, as follows.

The condition "I peaks" in the above definition of S_T indicates that I changes from *increasing* to *decreasing*. In terms of rates of change, this means I' changes from *positive* to *negative*. Now it stands to reason that, at a point where a quantity changes from being positive to being negative, it must equal *zero*. (This reasoning fails if the quantity in question has some kind of sudden "jump" from a positive to a negative value, but let's assume this is not the case. There's no reason to think I' would jump so abruptly.)

So: S_T is the value of S where $I' = 0$. But by the above SIR equations, $I' = aSI - bI$. So we see that S_T satisfies the equation

$$aS_T I - bI = 0.$$

Factoring out the I gives

$$I(aS_T - b) = 0.$$

Assuming $I \neq 0$, so we can divide both sides of this equation by I to get

$$aS_T - b = 0.$$

Solving for S_T gives our final formula for the threshold value of S:

$$\boxed{S_T = \frac{b}{a}.}$$

Threshold value S_T of S

For instance, in Example 2 above we have

$$S_T = \frac{0.2}{0.001} = 200.$$

In other words, as soon as the susceptible population has decreased from its initial value $S(0) = 500$ to just 200 remaining susceptible individuals, the disease (specifically, the infected population) reaches its peak, and thereafter starts to wane.

Remark 1.3.3. If the initial value $S(0)$ of S is *smaller* than the threshold value S_T, then I will decrease from the very outset (as long as it is nonzero to begin with). Indeed, suppose $S(0) < S_T$; that is, $S(0) < b/a$. Then, multiplying both sides by a, we get $aS(0) - b < 0$, so

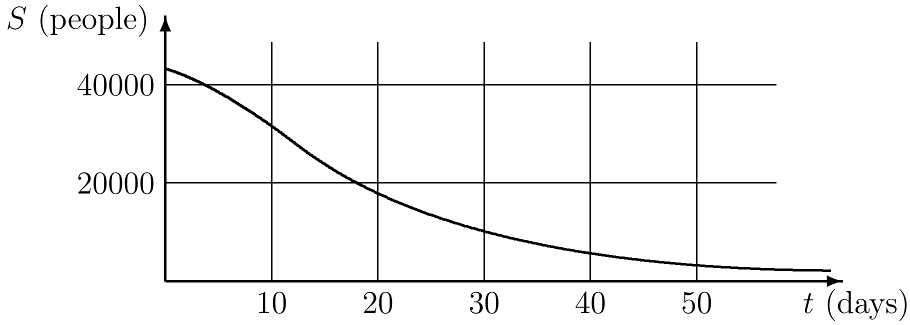
$$I'(0) = aS(0)I(0) - bI(0) = I(0)(aS(0) - b) < 0,$$

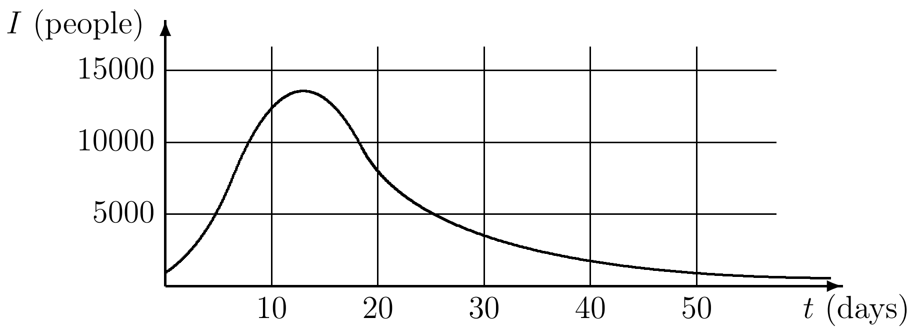
meaning I' is negative, and therefore I is decreasing, right at the beginning. In this case, the largest value of I occurs when $t = 0$; I only decreases from there.

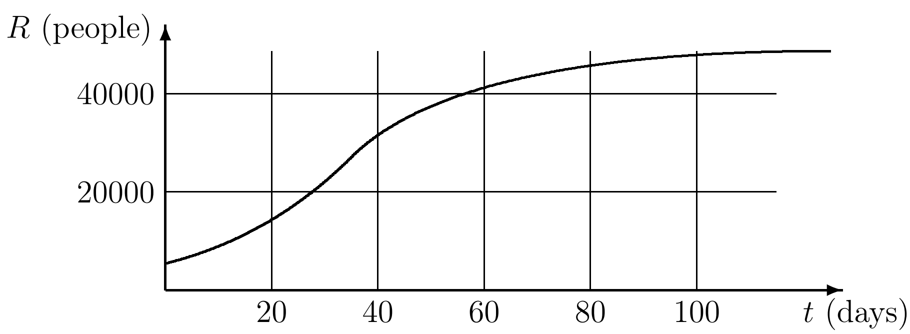
Exercises

Part 1: Reading a Graph (Exercises 1–6)

The graphs below have scales on the axes, so you can answer quantitative questions about them. For example, on day 20 there are about 18,000 susceptible people. Read the graphs to answer the following questions. (Note: $S + I + R$ is *not* constant in this example, so these graphs cannot be actual solutions to our model.)







1. When does the infection hit its peak? How many people are infected at that time?

2. Initially, how many people are susceptible? How many days does it take for the susceptible population to be cut in half?

3. How many days does it take for the recovered population to reach 25,000? How many people *eventually* recover? Where did you look to get this information?

4. On what day is the size of the infected population increasing most rapidly? When is it decreasing most rapidly? How do you know?

5. How many people caught the illness at some time during the first 20 days? (Note that this is not the same as the number of people who are infected on day 20.) Explain where you found this information. Hint: consider the graph of S.

6. Copy the graph of R as accurately as you can, and then superimpose a sketch of S on it. Notice the time scales on the *original* graphs of S and R are different. Describe what happened to the graph of S when you superimposed it on the graph of R. Did it get compressed or stretched? Was this change in the horizontal direction or the vertical?

Part 2: Mark Twain's Mississippi (Exercises 7–13)

The Lower Mississippi River meanders over its flat valley, forming broad loops called ox-bows. In a flood, the river can jump its banks and cut off one of these loops, getting shorter in the process. In his book *Life on the Mississippi* (1884), Mark Twain suggests, with tongue in cheek, that some day the river might even vanish! Here is a passage that shows us some of the pitfalls in using rates to predict the future and the past.

> In the space of one hundred and seventy six years the Lower Mississippi has shortened itself two hundred and forty-two miles. That is an average of a trifle over a mile and a third per year. Therefore, any calm person, who is not blind or idiotic, can see that in the Old Oölitic Silurian Period, just a million years ago next November, the Lower Mississippi was upwards of one million three hundred thousand miles long, and stuck out over the Gulf of Mexico like a fishing-pole. And by the same token any person can see that seven hundred and forty-two years from now the Lower Mississippi will be only a mile and three-quarters long, and Cairo [Illinois] and New Orleans will have joined their streets together and be plodding comfortably along under a single mayor and a mutual board of aldermen. There is something fascinating about science. One gets such wholesome returns of conjecture out of such a trifling investment of fact.

Let L be the length of the Lower Mississippi River. Then L is a variable quantity we shall analyze.

7. According to Twain's data, what is the exact **rate** at which L is changing, in miles per year? What approximation does he use for this rate? Is this a reasonable approximation? Is this rate *positive* or *negative*? Explain. In what follows, use Twain's approximation.

8. Twain wrote his book in 1884. Suppose the Mississippi that Twain wrote about had been 1100 miles long; how long would it have become in 1990?

9. Twain does not tell us how long the Lower Mississippi was in 1884 when he wrote the book, but he does say that 742 years later it will be only $1\frac{3}{4}$ miles long. How long must the river have been when he wrote the book?

10. Suppose t is the number of years since 1884. Write a formula that describes how much L has *changed* in t years. Your formula should complete the equation

$$\text{the change in } L \text{ in } t \text{ years} = \dots .$$

11. From your answer to question 9, you know how long the river was in 1884. From question 10, you know how much the length has changed t years after 1884. Now write a formula that describes how long the river is t years later.

12. Use your formula to find what L was a million years ago. Does your answer confirm Twain's assertion that the river was "upwards of 1,300,000 miles long" then?

13. Was the river ever 1,300,000 miles long; will it ever be $1\frac{3}{4}$ miles long? (This is called a **reality check**.) What, if anything, is wrong with the "trifling investment of fact" which led to such "wholesale returns of conjecture" that Twain has given us?

Part 3: A Measles Epidemic (Exercises 14–20)

We now consider a measles epidemic with transmission coefficient $a = 0.00001$ (person-day)$^{-1}$, and recovery coefficient $b = 1/14$ day^{-1}. This epidemic is then modeled by the equations

$$\begin{aligned} S' &= -0.00001\, SI, \\ I' &= 0.00001\, SI - I/14, \\ R' &= \phantom{-0.00001\, SI - {}} I/14. \end{aligned}$$

We assume that the initial values of S, I, and R are:

$$S(0) = 45{,}400, \quad I(0) = 2{,}100, \quad R(0) = 2{,}500.$$

14. How long does this disease last (that is, how long does one stay infected)? Hint: see the discussion of R' on page 12.

15. What is the threshold value S_T of S, for this epidemic?

16. Calculate the "current" rates of change $S'(0)$, $I'(0)$, and $R'(0)$, and use these rates of change to estimate $S(1)$, $I(1)$, and $R(1)$.

17. Using the values of $S(1)$, $I(1)$, and $R(1)$ found in the previous exercise, calculate $S'(1)$, $I'(1)$, and $R'(1)$, and use these rates of change to estimate $S(2)$, $I(2)$, and $R(2)$.

18. Using the values of $S(2)$, $I(2)$, and $R(2)$ found in the previous exercise, calculate $S'(2)$, $I'(2)$, and $R'(2)$, and use these rates of change to estimate $S(3)$, $I(3)$, and $R(3)$.

19. **Double the time step**. Go back to the starting time $t = 0$, and to the initial values

$$S(0) = 45{,}400, \qquad I(0) = 2{,}100, \qquad R(0) = 2{,}500.$$

Recalculate the values of S, I, and R at time $t = 2$, this time using stepsize $\Delta t = 2$. You should perform only a single round of calculations, using the rates $S'(0)$, $I'(0)$, and $R'(0)$.

20. **Quarantine**. For this exercise, you may wish to refer to the discussions of the proportions p and q, and of the transmission coefficient a, on page 13.

    One of the ways to treat an epidemic is to keep the infected away from the susceptible; this is called quarantine. The intention is to reduce the chance that the illness will be transmitted to a susceptible person. Thus, quarantine alters the *transmission coefficient*.

    (a) Suppose a quarantine is put into effect that cuts in half the chance that a susceptible will contact an infected. What is the new transmission coefficient?

    (b) Changing the transmission coefficient, as in part (a) of this exercise, changes the threshold level for S. What is the new threshold level S_T for this epidemic?

    (c) Does quarantine eliminate the epidemic, in the sense that the number of infected immediately goes down from 2,100, without ever showing an increase in the number of cases? (Assume, again, that we start with we start with $S(0) = 45{,}400$.) Hint: see Remark 1.3.3 above.

Part 4: Other Diseases (Exercises 21–22)

21. Suppose the spread of an illness similar to measles is modelled by the following rate equations:

$$\begin{aligned} S' &= -.00002\,SI, \\ I' &= .00002\,SI - .08\,I, \\ R' &= .08\,I. \end{aligned}$$

    Note: the initial values $S = 45{,}400$, etc. that we used above do not apply here.

    (a) Roughly how long does someone who catches this illness remain infected? Explain your reasoning. Hint: recall the discussion of R' in Section 1.2.

    (b) How large does the susceptible population have to be in order for the illness to take hold – that is, for the number of cases to increase? Explain your reasoning. Hint: see Remark 1.3.3 above.

    (c) Suppose 100 people in the population are currently ill. According to the model, how many (of the 100 infected) will recover during the next 24 hours?

    (d) Suppose 30 *new* cases appear during the same 24 hours. What does that tell us about S'?

    (e) Using the information in parts (c) and (d), can you determine how large the current susceptible population is?

22. (a) Construct the appropriate SIR model for a measles-like illness that lasts for 4 days. Assume it is also known that a typical susceptible person meets only about 0.3% of infected population each day, and the infection is transmitted in only one contact out of six. Hint: see the discussions of R' and S' in Section 1.2 above.

    (b) How small does the susceptible population have to be for this illness to fade away without becoming an epidemic? Hint: see Remark 1.3.3 above.

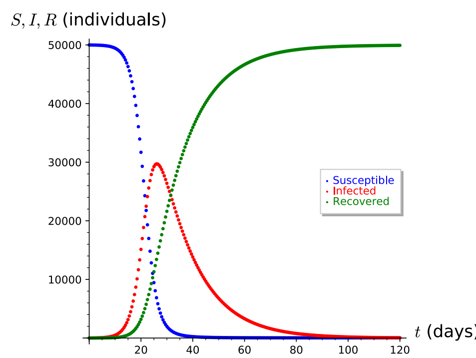
Part 5: SIR using Euler's method and Sage (Exercises 23–28)

For these exercises, you will study an SIR epidemic using Euler's method on a computer. You will do so by running, modifying, and thinking about the Sage worksheet SIR.sws, the code for which may be found in the Appendix at the end of this chapter.

Make sure you have uploaded the Sage worksheet SIR.sws to your Sage account. Your instructor will explain to you how to do this.

23. This exercise is meant simply to make sure your SIR program is working properly, and to get you thinking about coding in Sage.

    (a) Run your SIR program, by placing your cursor somewhere in the cell containing the code, and pressing Evaluate or Shift+Enter. Your graphical output should look something like this:

    For the rest of these exercises let's assume that, as in the above graph, our independent time variable t is measured in days, and that our dependent variables are measured in numbers of individuals.

    (b) What are our beginning and ending values of t? What is our stepsize? At how many total points in time will we make observations? Use the first five lines of your program to answer. (Hint: the first couple of lines of your program begin with "#," which means "This line is a *comment*, meant to explain to the reader what's going on.") Your answers should be *numbers*, like "23," not variable names like "tstart."

    (c) In this model, what are the transmission and recovery coefficients, and what are the initial values of S, I, and R? On average, how long does an individual remain infected? What is the threshold value S_T of S? Use the program to answer.

    (d) Write down the first three values of t (including $t = 0$) at which observations will be made and recorded. Also write down the last three.

    (e) What are the indented lines of your program doing? Describe what computations are being done, and how many times these computations are being executed. You may want to think about the flowchart in Figure 1.1 above.

    Estimate how long it would take you to do all of these calculations by hand (using a calculator that can only do $+$, $-$, $\times$, and $\div$). A VERY ROUGH ballpark estimate is fine, but do describe how you came up with that estimate.

(f) Why do you think the command used in the last lines of your program is called "list_plot" instead of just "plot"? Why couldn't we just use "plot" instead?

24. Run your SIR program again, but this time, with new stepsize=0.05 instead of the original stepsize found in the code (and all other quantities the same as above).

How is the output you got in this case different from that of Exercise 23 (besides the fact that the dots are more closely spaced in the second figure)? (It may help to look closely at, among other things, where I peaks in each of your two figures. The difference is a bit subtle, but it's there. Zoom in on your graphs if necessary.)

Explain why the two graphs should look different. Which of the two figures do you think is "better," in the sense of giving a closer approximation to reality?

From now on, we will use a stepsize of 0.05.

25. Run your SIR program again, but this time, with $b = 1/28$ instead of $b = 1/14$ (and all other quantities the same as in Exercise 24 above).

(a) What are the changes in the graphs of S, I, and R, relative to the graphs in Exercise 24? Describe in general terms; you don't have to discuss specific numerical values, although you can if you want.

(b) From a modeling perspective (that is, in terms of the "real life" interpretation), what's the meaning of the recovery coefficient $b = 1/28$? Explain, from a modeling perspective, why it makes sense that changing b from $1/14$ to $1/28$ would cause changes like the ones you saw in the graphs of S, I, and R.

(c) What is your new threshold value S_T? How does this compare to your answer from Exercise 23(c)? That is, which of these threshold values is larger, and by how much? Explain why this makes sense, from a modeling/"real-world" perspective.

26. **Quarantine and flattening the curve.** Reset b to the value $b = 1/14$ of Exercise 24.

The effect of quarantine – that is, isolating infected individuals from the general population – is to decrease the likelihood of contact of a susceptible individual with an infected individual. (Social distancing, for example, is a form of quarantine.)

(a) Suppose a quarantine is imposed that cuts this likelihood in half. Which of the two parameters, a or b, will change as a result of this, and by how much will it change? Hint: you may want to refer to the discussions of p, q, and a on page 13 above.

Once you have figured out the answer, make the corresponding change to your SIR code, and run the program again to generate a new graph.

(b) Describe how this change affects your I curve, relative to the earlier graph of I in Exercise 24. In particular, what happens to the peak of the I curve? Does this peak happen sooner, or later, than it did before? Is this peak higher, or lower, than it was before?

(c) Describe how this change affects *the total number of people who get infected over the course of the disease.* Hint: don't look at the I curve for this, because the I curve represents the number infected on a given day, and its hard to deduce from this how many become infected in total. Instead, look at the S curve. The number of susceptibles at the outset, minus the number of susceptibles at the end, tells you how many became infected over the course of the disease. (Assume that the disease has pretty much stabilized – the graphs won't change much – after 120 days.)

(d) What does quarantine seem to affect more dramatically: the number of people who ultimately become ill, how long it takes before the illness peaks, or the maximum number of people who can become ill at the same time? (Your answer may include more than one of these three phenomena.)

27. Reset all parameters to the values of Exercise 24. (So stepsize=0.05, $a = 0.00001$, $b = 1/14$.) We are now going to tweak SIR so that recovered become *susceptible again* after 10 days. This is sometimes called the $SIRS$ model – the idea is that, in this case, immunity doesn't last forever, so the "R" population feeds back into the "S" population.

    To do this:

    (a) Make the appropriate changes to the program, to reflect this new phenomenon where recovered become susceptible again. You should only need to change *two lines* of code in your program to do this. Then execute the new code.

    (b) Explain what changes you made to your SIR code to get your new program. You can do this by describing these changes in a brief paragraph, or by just specifying which lines you changed, and writing down what you changed them to.

    (c) What are the changes in the graphs of S, I, and R, relative to the graphs in Exercise 24? Describe in general terms; you don't have to discuss specific numerical values, although you can if you want. Explain, from a modeling perspective, why it makes sense that the changes you made to the code would cause changes like the ones you saw in the graphs of S, I, and R.

28. Notice that, in the graph you generated in Exercise 27 above, values of I level off at a higher level than values of R. What *single parameter* would you change, and to what would you change it, to make I and R level off at the *same* height? (You might change a certain parameter to make I level off at the height of R; or you might change a certain parameter to make R level off at the height of I. Either way is fine.) Explain why this makes sense from a modeling perspective.

    Once you've figured out how to answer the above question, make the required changes to your program, and run it.