Taylor polynomials and series.

A) Recap: Taylor polynomials.

Suppose f is a function that is n times differentiable at $x=a$ – that is, $f(a), f'(a), f''(a), \ldots, f^{(n)}(a)$ all exist. Then we define the n^{th} degree Taylor polynomial for $f(x)$ at $x=a$, denoted $T_n(x)$, by

$$T_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3$$

$$+ \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k. \qquad (*)$$

Then, from $(*)$, we see that $T_n(x)$ satisfies:

(0) $T_n(a) = f(a) + \{\text{terms with a factor of } (a-a)\}$
$= f(a) + 0 = f(a).$

(1) $T_n'(x) = f'(a) + \{\text{terms with a factor of } (x-a)\}$,
so
$T_n'(a) = f'(a) + \{\text{terms with a factor of } (a-a)\}$
$= f'(a) + 0 = f'(a).$

(2) $T_n''(x) = f''(a) + \{\text{terms with a factor of } (x-a)\}$,
so
$T_n''(a) = f''(a) + \{\text{terms with a factor of } (a-a)\}$
$= f''(a) + 0 = f''(a).$

...

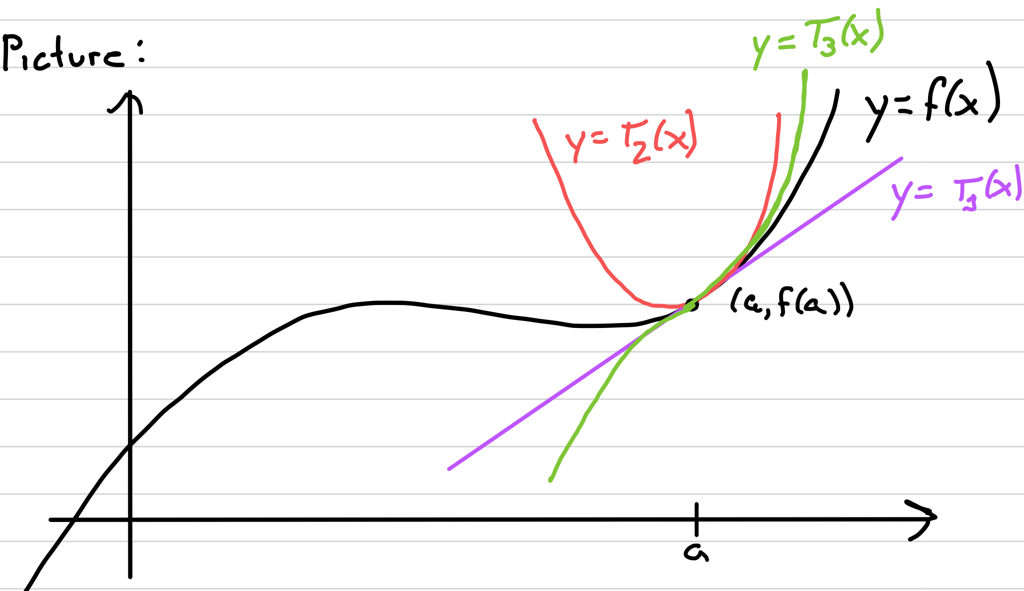
(n) $T_n^{(n)}(x) = f^{(n)}(a) + \{\text{terms with a factor of } (x-a)\}$,

so
$$T_n^{(n)}(a) = f^{(n)}(a) + \{\text{terms with a factor of } (a-a)\}$$
$$= f^{(n)}(a) + 0 = f^{(n)}(a).$$

In sum: the $k^{\underline{th}}$ derivative of T_n and the $k^{\underline{th}}$ derivative of f agree, for $0 \le k \le n$.

The BIG IDEA here is this:
Because derivatives dictate "shape," T_n should look a lot like f, near the point $x=a$.

Picture:



(But T_n, being a polynomial, is generally simpler than f algebraically.)

Example:

we compute the 3rd degree Taylor polynomial T_3 for

$$f(x)=\frac{1}{\sqrt{5-x}} \quad \text{at} \quad x=1.$$

n	0	1	2	3
$f^{(n)}(x)$	$\frac{1}{\sqrt{5-x}}$	$\frac{1/2}{(5-x)^{3/2}}$	$\frac{3/4}{(5-x)^{5/2}}$	$\frac{15/8}{(5-x)^{7/2}}$
$f^{(n)}(1)$	$1/2$	$1/16$	$3/128$	$15/1024.$

So

$$T_n(x)=\frac{1}{2}+\frac{1}{16}(x-1)+\frac{3}{128\cdot 2!}(x-1)^2+\frac{15}{1024\cdot 3!}(x-1)^3$$

$$=\frac{1}{2}+\frac{1}{16}(x-1)+\frac{3}{256}(x-1)^2+\frac{5}{2048}(x-1)^3.$$

For example, since $x=1.1$ is "close" to $x=1$, we approximate:

$$f(1.1)\approx T_3(1.1)=\frac{1}{2}+\frac{0.1}{16}+\frac{0.03}{256}+\frac{0.005}{2048}$$

$$=0.506369628\ldots$$

["True" value: $f(1.1)=0.506369683\ldots$]

B) Taylor series.

If $f^{(k)}(a)$ exists for all $k=0,1,2,3,\ldots$, then we can define the Taylor series $T(x)$ by

$$T(x) = \lim_{n\to\infty} T_n(x) = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

Also: if $a=0$, we call the Taylor series a Maclaurin series.

Example: we've seen that $f(x)=\sin x$ has Maclaurin series

$$T(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}.$$

We haven't seen, yet, whether $T(x)$ converges to $f(x)$.

An easy ratio test shows that the Taylor series $T(x)$ for $f(x)=\sin x$ converges, for all x. (DIY).

But this does not prove that $T(x)$ converges to $f(x)$!!
For that, we'll need more (to be seen soon).