Power series.

Recall: if $f(x)$ is a function such that $f^{(n)}(a)$ exists for $0 \leq n \leq N$, then $f(x)$ has an N^{th} degree Taylor polynomial

$$a_0 + a_1(x-a) + a_2(x-a)^2 + \ldots + a_N(x-a)^N = \sum_{n=0}^{N} a_n (x-a)^n, \qquad (*)$$

where $a_n = f^{(n)}(a)/n!$ for all n.

We want to let $n \to \infty$ in $(*)$, and consider series like

$$\sum_{n=0}^{\infty} a_n (x-a)^n = a_0 + a_1(x-a) + \ldots + a_n(x-a)^n + \ldots \qquad (**)$$

We call such series (whatever the a_n's may be) power series.

TODAY'S BIG QUESTION:
Where (for which values of x) does $(**)$ converge?

The ANSWER lies in the ratio test.

Example 1. Discuss convergence of

$$\sum_{n=1}^{\infty} \frac{(x-2)^n}{n \cdot 3^n}.$$

Solution.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}(x-2)^{n+1}}{a_n(x-2)^n}\right| = \lim_{n\to\infty}\left|\frac{(x-2)^{n+1}}{(n+1)\cdot 3^{n+1}}\cdot\frac{n\cdot 3^n}{(x-2)^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(x-2)}{3}\cdot\frac{n}{n+1}\right| = \frac{|x-2|}{3}\lim_{n\to\infty}\frac{n}{n+1}$$

$$= \frac{|x-2|}{3}.$$

So the series converges absolutely if:

$$\frac{|x-2|}{3} < 1$$

$$|x-2| < 3$$

$$-3 < x-2 < 3$$

$$-1 < x < 5.$$

It diverges if $\frac{|x-2|}{3} > 1$, i.e. $x < -1$ or $x > 5$.

We check the endpoints $x=-1$ and $x=5$ separately:

(a) $x=-1$: $\displaystyle\sum_{n=1}^{\infty}\frac{(x-2)^n}{n\cdot 3^n} = \sum_{n=1}^{\infty}\frac{(-1-2)^n}{n\cdot 3^n} = \sum_{n=1}^{\infty}\frac{(-3)^n}{n\cdot 3^n}$

$\displaystyle = \sum_{n=1}^{\infty}\frac{(-1)^n}{n}$: converges conditionally.

(b) $x=5$: $\displaystyle\sum_{n=1}^{\infty}\frac{(x-2)^n}{n\cdot 3^n} = \sum_{n=1}^{\infty}\frac{(5-2)^n}{n\cdot 3^n} = \sum_{n=1}^{\infty}\frac{3^n}{n\cdot 3^n}$

$\displaystyle = \sum_{n=1}^{\infty}\frac{1}{n}$: diverges.

SUMMARY: the series converges absolutely if $-1 < x < 5$. It converges if $-1 \leq x < 5$, and diverges otherwise.

We call:

- $[-1, 5)$ the <u>interval of convergence</u> (IOC) of the series;
- half the length of IOC the <u>radius of convergence</u> of the series ($= 3$ in this case)

Example 1 illustrates:

THEOREM (power series convergence).

Consider the series

$$\sum_{n=m}^{\infty} a_n (x-a)^n$$

(m can be any integer). Suppose

$$\lim_{n\to\infty} \left| \frac{a_{n+1}(x-a)^{n+1}}{a_n (x-a)^n} \right| = |x-a| \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = |x-a| \cdot L$$

for some number L, with $0 \le L \le \infty$. Then:

(a) The series converges absolutely for $|x-a| L < 1$.
(b) The series diverges for $|x-a| \cdot L > 1$.
(c) The points where $|x-a| \cdot L = 1$ must be checked separately.
(d) The ROC of the series is $1/L$.
(e) $L = 0 \Rightarrow$ absolute convergence for all x.
(f) $L = \infty \Rightarrow$ convergence only at $x = a$.

Examples: discuss convergence.

2) $\sum_{n=0}^{\infty} \frac{x^n}{n!}$.

Solution.

$$\lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| = |x| \lim_{n\to\infty} \frac{1}{n+1} = 0:$$

absolute convergence for all x.

IOC: $(-\infty,\infty)$.

ROC: $+\infty$.

3) $\sum_{n=1}^{\infty} \frac{(-1)^n (x-1)^n}{n}$.

Solution.

$$\lim_{n\to\infty} \left| \frac{(-1)^{n+1} (x-1)^{n+1}}{n+1} \cdot \frac{n}{(-1)^n (x-1)^n} \right|$$

$$= \lim_{n\to\infty} \left| (-1)(x-1) \cdot \frac{n}{n+1} \right| = |x-1| \lim_{n\to\infty} \frac{n}{n+1}$$

$$= |x-1| .$$

Converges absolutely if:

$$|x-1| < 1$$

$$-1 < x-1 < 1$$

$$0 < x < 2.$$

Check endpoints:

(i) $x=0$: $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}(0-1)^n = \sum_{n=1}^{\infty} \frac{1}{n}$: diverges

(ii) $x=2$: $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}(2-1)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$: converges conditionally

IOC: $(0,2]$. ROC: 1.