Homework #9: Due 1 PM Thursday, April 16

1. If $\sum b_n(x-2)^n$ converges at $x=0$ but diverges at $x=7$, what is the largest possible interval of convergence of this series? What's the smallest possible? Largest: $[-3,7)$. Smallest: $[0,4)$.

2. (a) Write down the second degree Taylor polynomial $P_2(x)$ approximating

$$f(x) = \ln\big(1+x(1-x)\big)$$

near $x=0$.

$P_2(x) = x - \dfrac{3x^2}{2}$.

(b) Use your result from part (a) to approximate $\ln(1.09)$. Hint: $x=0.1$.

$\ln(1.09) = f(0.1) \approx P_2(0.1) = 0.085$.

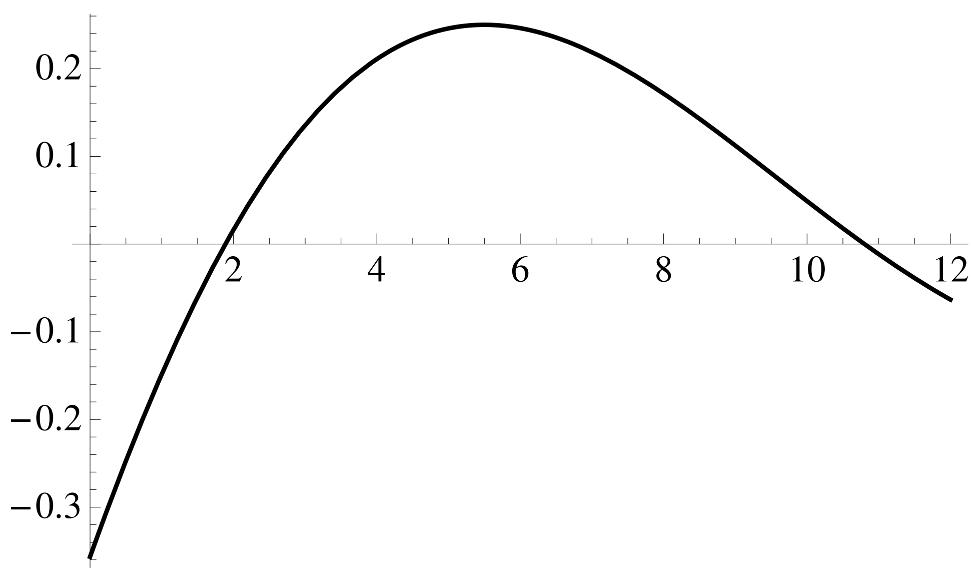
(c) What does Taylor's inequality say about the error in the approximation you found in part (b)? You should find it useful to note that

$$f'''(x) = \frac{2(2x-1)\left(x^2-x+4\right)}{\left(x^2-x-1\right)^3},$$

and that $f'''(x)$ is a *decreasing* function on the interval $(0, 0.1)$.

$$|R_2(x)| \le \frac{|f'''(0)|\,|0.1|^3}{3!} = \frac{|2(-1)(4)|\,|0.1|^3}{3!} = 0.00133.$$

3. Consider the function $y=f(x)$ sketched below.



Suppose $f(x)$ has Taylor series

$$f(x) = a_0 + a_1(x-4) + a_2(x-4)^2 + a_3(x-4)^3 + \ldots$$

about $x=4$.

(a) Is a_0 positive or negative? Please explain. $a_0 > 0$, because the function is positive at $x = 4$.

(b) Is a_1 positive or negative? Please explain. $a_1 > 0$, because the function is increasing at $x = 4$.

(c) Is a_2 positive or negative? Please explain. $a_2 < 0$, because the function is concave down at $x = 4$.

4. How many terms of the Taylor series for $\ln(1 + x)$ centered at $x = 0$ do you need to estimate the value of $\ln(1.4)$ to three decimal places?

   We will use the error bound. The error bound corresponding to $T_n(0.4)$ is given by $\frac{M(0.4)^{n+1}}{(n+1)!}$, where M is the maximum of $|f^{n+1}(u)|$ on the interval $[0, 0.4]$.

   For $n \geq 1$, the derivatives of $f(x) = \ln(1 + x)$ are given by the following formula: $f^n(x) = \frac{(-1)^{n-1}(n-1)!}{(1+x)^n}$ Clearly, $|f^{n+1}(u)| = \frac{n!}{(1+u)^{n+1}}$ is decreasing on the interval $[0, 0.4]$, so $M = \frac{n!}{(1+0)^{n+1}} = n!$ The error bound is then $\frac{n!(0.4)^{n+1}}{(n+1)!} = \frac{(0.4)^{n+1}}{n+1}$. The first n for which the error bound is smaller than 0.0005 is $n = 6$.

5. Find the integral and express the answer as an infinite series.

$$\int \frac{e^x - 1}{x}\, dx = \sum_{n=1}^{\infty} \frac{x^n}{n \cdot n!} + C$$

6. Using series, evaluate the limit

$$\lim_{x\to 0} \frac{\sin x - x}{x^3} = -\frac{1}{6}.$$

7. Use Taylor's inequality for $P_n(x)$ to find a reasonable bound for the error in approximating the quantity $e^{0.60}$ with a third degree Taylor polynomial for e^x centered at $x = 0$.

   For $n = 3$, the error bound is given by $\frac{Mx^4}{4!}$, where M is the maximum of $|f^4(u)| = |e^u|$ on the interval between 0 and x. For $x = 0.6$, M is the maximum of e^y (an increasing function) on the interval $[0, 0.6]$, so $M = e^{0.6}$. The bound is: $\frac{e^{0.6}(0.6)^4}{4!}$.

8. Consider the error in using the approximation $\sin\theta \approx \theta - \theta^3/3!$ on the interval $[-1, 1]$. Where is the approximation an overestimate? Where is it an underestimate?

   For $0 \leq \theta \leq 1$, the estimate is an underestimate (the alternating Taylor series for $\sin\theta$ is truncated after a negative term). For $-1 \leq \theta \leq 0$, the estimate is an overestimate (the alternating Taylor series is truncated after a positive term).

9. Find the Taylor series around $x = 0$ for

$$\cosh x = \frac{e^x + e^{-x}}{2}.$$

(Your answer should involve only even powers of x.)

$$\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}.$$

10. Suppose the series

$$\sum_{n=5}^{\infty} C_n (x-3)^n$$

converges when $x = 0$ but diverges when $x = 9$. For each of the following values of x, determine whether the series converges or diverges there, or if there's not enough information to say. Explain each of your answers briefly.

(a) $x = 1$ Converges: since the series converges at $x = 0$, which is 3 units away from the center $a = 3$, it converges at least on the interval $[0, 6)$.

(b) $x = -5$ Diverges: since the series diverges at $x = 9$, which is 6 units away from the center $a = 3$, it diverges at least on the interval $(-\infty, -3)$ and on $[9, \infty)$.

(c) $x = -2$ Not enough information, since $x = -2$ is neither within 3 units of the center nor beyond 6 units from the center.