Taylor's Theorem: recap/rehash/remix.

The Theorem says: let $R_n(x)$ be the difference between a function $f(x)$ and its n^{th} degree Taylor polynomial at $x=a$:

$$R_n(x) = f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k. \qquad (*)$$

Then:

(i) Suppose $\lim_{n\to\infty} R_n(x) = 0$. Then by $(*)$,

$$0 = f(x) - \lim_{n\to\infty} \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k,$$

or in other words,

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

($f(x)$ equals its Taylor series at $x=a$).

(ii) Let d be any number. Then

$$|R_n(d)| \le \frac{M|d-a|^{n+1}}{(n+1)!},$$

where M is any number such that $|f^{(n+1)}(x)| \le M$ for all x between a and d.

Example:

(a) Compute the Maclaurin series $T(x)$ for

$$f(x) = \frac{1}{\sqrt{1-x}}.$$

(b) Find the interval of convergence of this Maclaurin series.

(c) How many terms in this series are required to approximate

$$\frac{1}{\sqrt{1.01}}$$

to within an error of 10^{-7}?

Solution.

(a) We build a table:

k	0	1	2	3	4	$\cdots$
$f^{(k)}(x)$	$\frac{1}{\sqrt{1-x}}$	$\frac{1}{2(1-x)^{3/2}}$	$\frac{3}{4(1-x)^{5/2}}$	$\frac{3\cdot 5}{8(1-x)^{7/2}}$	$\frac{3\cdot 5\cdot 7}{16(1-x)^{9/2}}$	
$f^{(k)}(0)$	1	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{3\cdot 5}{2^3}$	$\frac{1\cdot 3\cdot 5\cdot 7}{2^4}$	

We can see that

$$T(x) = 1 + \sum_{k=1}^{\infty} \frac{1\cdot 3\cdot 5\cdot 7\cdots(2k-1)}{2^k\, k!}\, x^k.$$

(b) $\lim_{n\to\infty}\left|\frac{(n+1)^{st} \text{ term}}{n^{th} \text{ term}}\right|$

$$= \lim_{n\to\infty}\left|\frac{1\cdot3\cdot5\cdot7\cdots(2(n+1)-1)}{2^{n+1}(n+1)!}x^{n+1}\cdot\frac{2^n n!}{1\cdot3\cdot5\cdot7\cdots(2n-1)x^n}\right|$$

$$= |x|\lim_{n\to\infty}\frac{(2(n+1)-1)}{2(n+1)} = |x|\lim_{n\to\infty}\frac{2n+1}{2n+2} = |x|.$$

So the series converges absolutely for $|x|<1$.

One checks (this is not easy!) that the series converges at $x=-1$ but diverges at $x=1$.
So the interval of convergence is $[-1,1)$.

[It's harder to show - but it's true - that this series converges to $f(x)$ there.]

(c) We want to approximate

$$\frac{1}{\sqrt{1.01}} = \frac{1}{\sqrt{1-(-.01)}} = f(-.01)$$

by a Taylor polynomial $T_n(-0.01)$. By part (ii) of our Theorem, we know that the error $R_n(-0.01)$ satisfies

$$|R_n(-0.01)| \le \frac{M|-.01|^{n+1}}{(n+1)!}, \qquad (**)$$

where M is any number with $|f^{(n+1)}(x)| \le M$ for all x between 0 and -0.01.

But we compute that

$$f^{(n+1)}(x) = \frac{1\cdot 3\cdot 5\cdot 7\cdots(2n+1)}{2^{n+1}(1-x)^{(n+1)/2}}.$$

The largest this can be, for x between 0 and -0.01, is when x is largest, meaning $x=0$. So

$$|f^{(n+1)}(x)| \le \frac{1\cdot 3\cdot 5\cdot 7\cdots(2n+1)}{2^{n+1}}.$$

So by $(**)$,

$$|R_n(-0.01)| \le \frac{1\cdot 3\cdot 5\cdot 7\cdots(2n+1)(.01)^{n+1}}{2^{n+1}(n+1)!}$$

We check, by plugging in, that the right side becomes $< 10^{-7}$ as soon as $n = 3$. So the Taylor polynomial

$$T_3(x) = 1 + \frac{x}{2} + \frac{3x^2}{4 \cdot 2!} + \frac{3 \cdot 5x^3}{8 \cdot 3!}$$

will suffice for our purposes.