# Taylor series: Miscellaneous examples and applications.

## Example 1: a Maclaurin series for $\cos x$.

We've seen that

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

for all $x$. Differentiate term-by-term to get

$$\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \cdot (2k+1) x^{2k} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \quad \text{for all } x.$$

## Example 2.

A long long time ago, in class, we saw that

$$f(x) = e^x$$

has seventh degree Taylor polynomial

$$T_7(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^7}{7!}.$$

FACT: the pattern continues. That is,

$$e^x = \frac{1}{0!} + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots = \sum_{k=0}^{\infty} \frac{x^k}{k!} \quad \text{for all } x.$$

Example 3: the generalized binomial theorem.

Let
$$f(x) = (1+x)^b$$

where $b$ is any real number. Let's do some derivatives:

| $k$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $f^{(k)}(x)$ | $(1+x)^b$ | $b(1+x)^{b-1}$ | $b(b-1)(1+x)^{b-2}$ | $b(b-1)(b-2)(1+x)^{b-3}$ |
| $f^{(k)}(0)$ | $1$ | $b$ | $b(b-1)$ | $b(b-1)(b-2)$ |

The pattern is clear: $f^{(k)}(0) = b(b-1)\cdots(b-(k-1))$.

So we have the Maclaurin series
$$\sum_{k=0}^{\infty} \frac{b(b-1)(b-2)\cdots(b-(k-1))}{k!} x^k.$$

One shows that this series equals $f(x)$ for $|x|<1$ (and sometimes, depending on $b$, for $x=-1$ and/or $x=1$).

## Comments:

A) It's common to write $\binom{b}{k}$ ("b choose k") for $\frac{b(b-1)(b-2)\cdots(b-(k-1))}{k!}$. So we can write

$$(1+x)^b = \sum_{k=0}^{\infty} \binom{b}{k} x^k \qquad \text{for } |x|<1.$$

B) All of this gives the usual binomial theorem if $b$ is a positive integer (though this is not obvious).

## Example 3.

The "sinc function"

$$\operatorname{sinc}(x) = \frac{\sin x}{x}$$

has no simple, compact antiderivative. But we can still integrate it! Here's how: by the Taylor series for $\sin x$, we have

$$\operatorname{sinc}(x) = \frac{\sin x}{x} = \frac{1}{x}\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k},$$

for all $x$. Then

$$\int \operatorname{sinc}(x)\,dx = \int \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k}\,dx$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \int x^{2k}\,dx$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)! \cdot (2k+1)} x^{2k+1} + C.$$

Example 4: using series to compute limits.

We could find a limit like

$$\lim_{x \to 0} \frac{\cos x - 1 + \frac{x^2}{2}}{x^4}$$

using l'Hôpital's rule. Or we could use series, like this:

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + (\text{higher powers of } x),$$

so

$$\cos x - 1 + \frac{x^2}{2} = \frac{x^4}{4!} - \frac{x^6}{6!} + (\text{higher powers of } x),$$

so

$$\frac{\cos x - 1 + \frac{x^2}{2}}{x^4} = \frac{1}{4!} - \frac{x^2}{6!} + (\text{higher powers of } x),$$

so

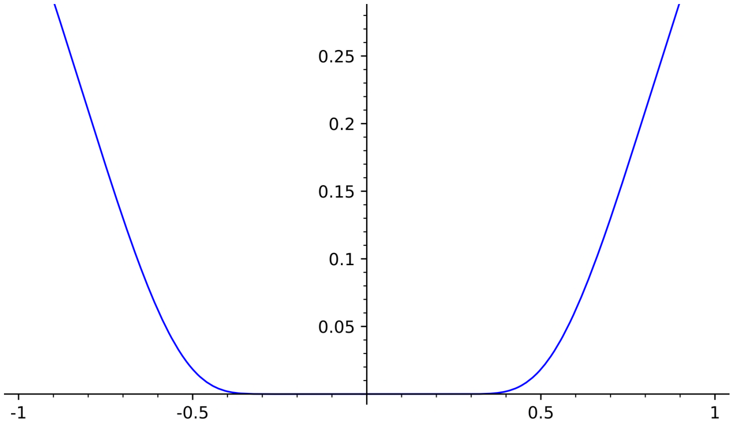
$$\lim_{x \to 0} \frac{\cos x - 1 + \frac{x^2}{2}}{x^4} = \frac{1}{4!} - \frac{0^2}{6!} + (\text{higher powers of } 0)$$

$$= \frac{1}{4!} = \frac{1}{24}.$$

**Example 5:** A function $f$ whose Maclaurin series converges, but not to $f$.

Let $f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$

The graph of $f$ looks like this:




One shows: $f^{(k)}(0) = 0$ for all $k$: so the Maclaurin series

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = \sum_{k=0}^{\infty} \frac{0}{k!} x^k = \sum_{k=0}^{\infty} 0 = 0$$

converges (to zero) for all $x$. But $f(x) = 0$ only at $x = 0$. So the Maclaurin series, although it converges everywhere, converges to $f(x)$ only at $x = 0$!!