BRIEF intro to Fourier analysis.

GOAL: to investigate Fourier's 1807 claim:

"there is no function... which cannot be expressed by a trigonometric series."

Part I: complex numbers.

Throughout, x, x_1, x_2, y, u, v denote arbitrary real numbers.

Definition 1.

(a) Let i denote a square root of -1, so $i^2 = 1$. A complex number is a quantity $x+iy$. The set of all complex numbers is denoted $\mathbb{C}$.

Geometrically, we think of $\mathbb{C}$ as the usual xy-plane $\mathbb{R}^2$: that is, we identify $x+iy \in \mathbb{C}$ with the point $(x,y) \in \mathbb{R}^2$. In particular, since $i = 0+1i$, we identify i with $(0,1)$.

(b) Let $z = x+iy \in \mathbb{C}$. Then:

- We call x the real part of z, denoted $\operatorname{Re} z$, and call y the imaginary part of z, denoted $\operatorname{Im} z$.

- The modulus $|z|$ is the distance from z to $(0,0)$. So $|z| = \sqrt{x^2+y^2}$.

Also, the angle that z makes with the positive x-axis is called the argument of z, denoted $\operatorname{Arg} z$. We assume $\operatorname{Arg} z \in (-\pi, \pi]$.

Finally, we denote by $\bar{z}$ the reflection of z about the x-axis. So $\bar{z} = x - iy$. We call $\bar{z}$ the complex conjugate of z.

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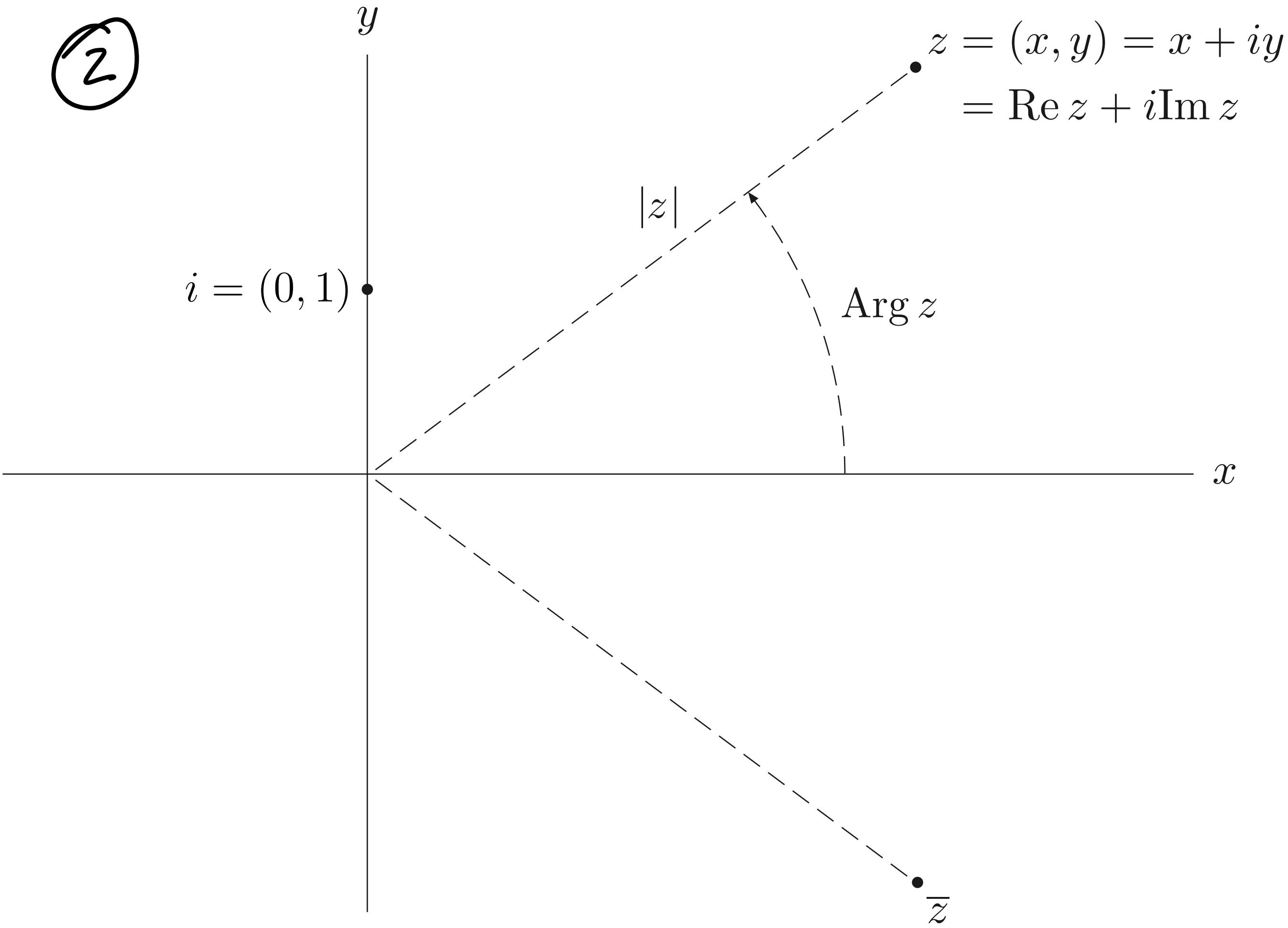


y
$z = (x, y) = x + iy$
$= \mathrm{Re}\, z + i \mathrm{Im}\, z$
$|z|$
$i = (0, 1)$
$\mathrm{Arg}\, z$
x
$\overline{z}$

(c) We define addition and multiplication in $\mathbb{C}$ by way of the operations on $\mathbb{R}$, keeping in mind that $i^2 = -1$:

$$(x+iy)+(u+iv) = (x+u)+i(y+v);$$
$$(x+iy)(u+iv) = xu + x(iv) + iyu + (iy)(iv)$$
$$= (xu - yv) + i(xv + yu).$$

Exercise 1: Using the above definitions, show that

(a) $z + \bar{z} = 2\,\mathrm{Re}\, z$; $\quad z - \bar{z} = 2i\,\mathrm{Im}\, z$;

(b) $z\bar{z} = |z|^2$;

(c) If $z \neq 0$ then the unique complex number z^{-1} (also written $1/z$) such that $zz^{-1} = z^{-1}z = 1$ is given by $z^{-1} = (x - iy)/(x^2+y^2)$.

Part II: the function e^{ix}.

Definition: if $x \in \mathbb{R}$, we define the complex exponential function e^{ix} by

$$e^{ix} = \cos x + i \sin x.$$

Theorem 1: properties of e^{ix}.

We have:

(a) $|e^{ix}| = 1$.

(b) $e^{ix_1} e^{ix_2} = e^{i(x_1+x_2)}$

(c) $1/e^{ix} = e^{-ix}$ (where e^{-ix} means $e^{i(-x)}$).

(d) $(e^{ix})^n = e^{inx}$ for $n \in \mathbb{N}$.

(e) $e^{in\pi} = (-1)^n$ for $n \in \mathbb{N}$.

Proof:

(a) $|e^{ix}| = |\cos x + i \sin x| = \sqrt{\cos^2 x + \sin^2 x} = 1$.

(b) $e^{ix_1} e^{ix_2} = (\cos x_1 + i \sin x_1)(\cos x_2 + i \sin x_2)$

$$= \cos x_1 \cos x_2 + i \sin x_1 \cos x_2 + i \cos x_1 \sin x_2 + i^2 \sin x_1 \sin x_2$$

$$= (\cos x_1 \cos x_2 - \sin x_1 \sin x_2) + i(\sin x_1 \cos x_2 + \cos x_1 \sin x_2)$$

$$= \cos(x_1+x_2) + i \sin(x_1+x_2)$$

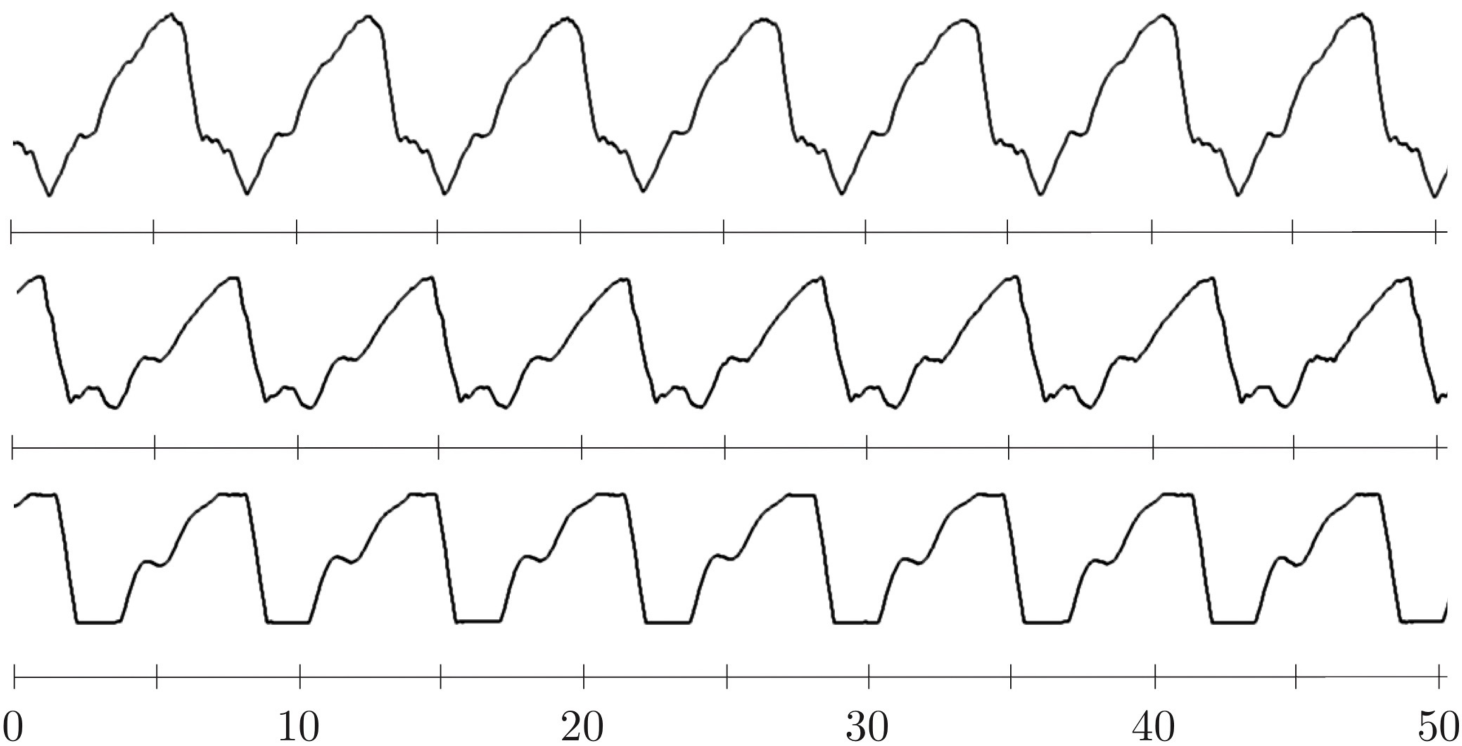
trig identities

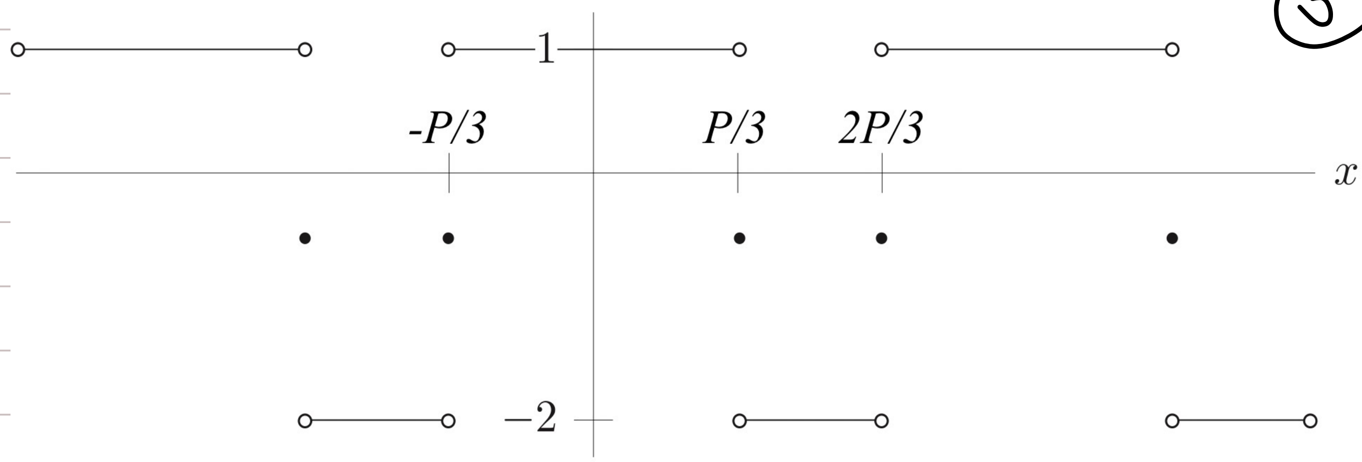
$$= e^{i(x_1+x_2)}.$$

Parts (c)(d)(e): this is Exercise 2. $\square$

Part III: periodic functions.

Definition 2. Let $P>0$.
A function $f: \mathbb{R} \to \mathbb{R}$ is said to be <u>P-periodic</u> if $f(x+P) = f(x)$ $\forall\, x \in \mathbb{R}$.
A P-periodic function is one whose graph repeats itself every P units.




1
-P/3
P/3
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x
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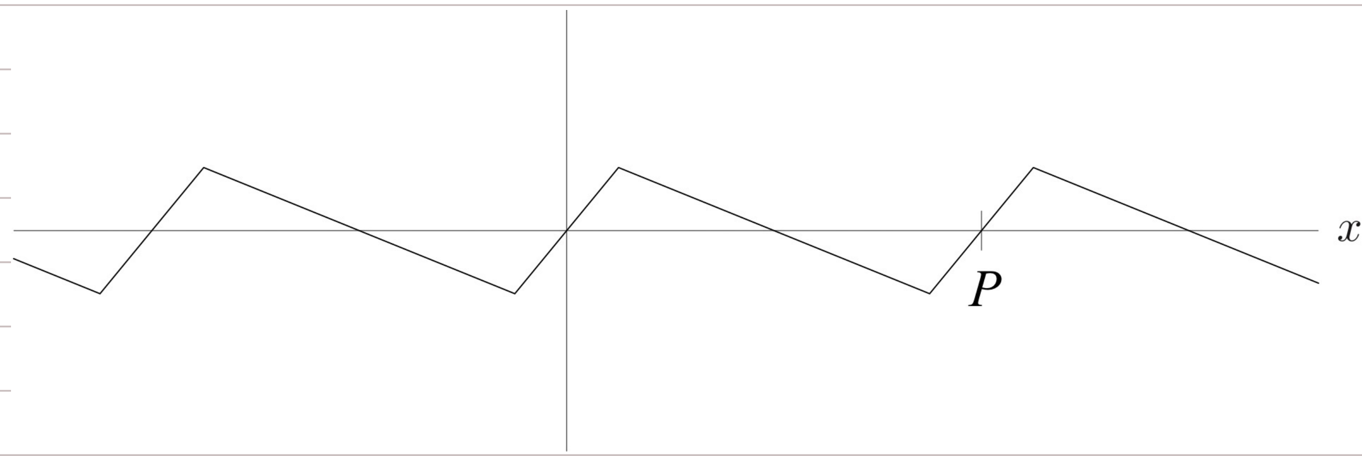
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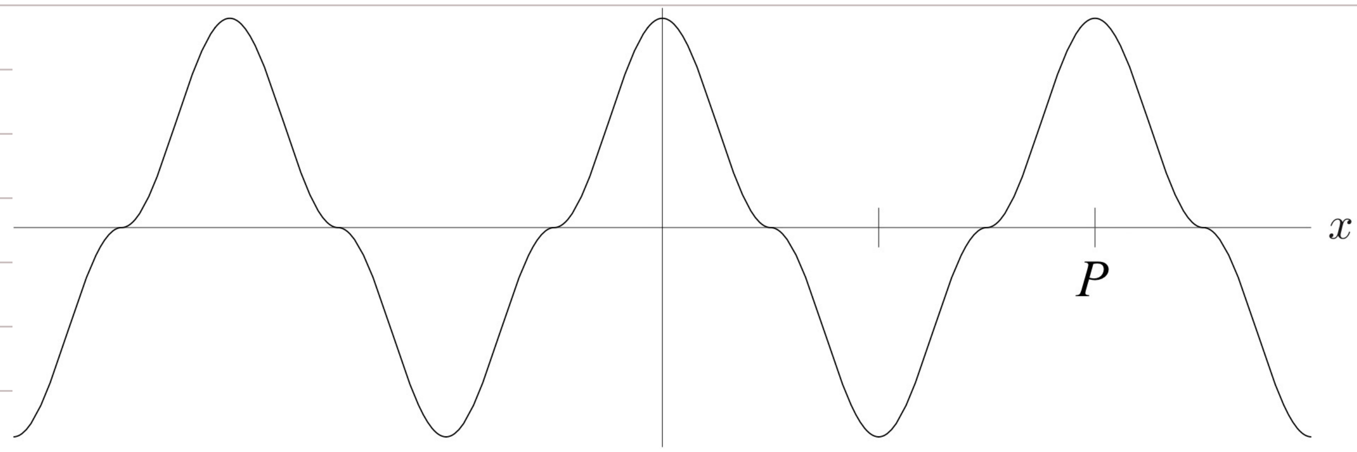
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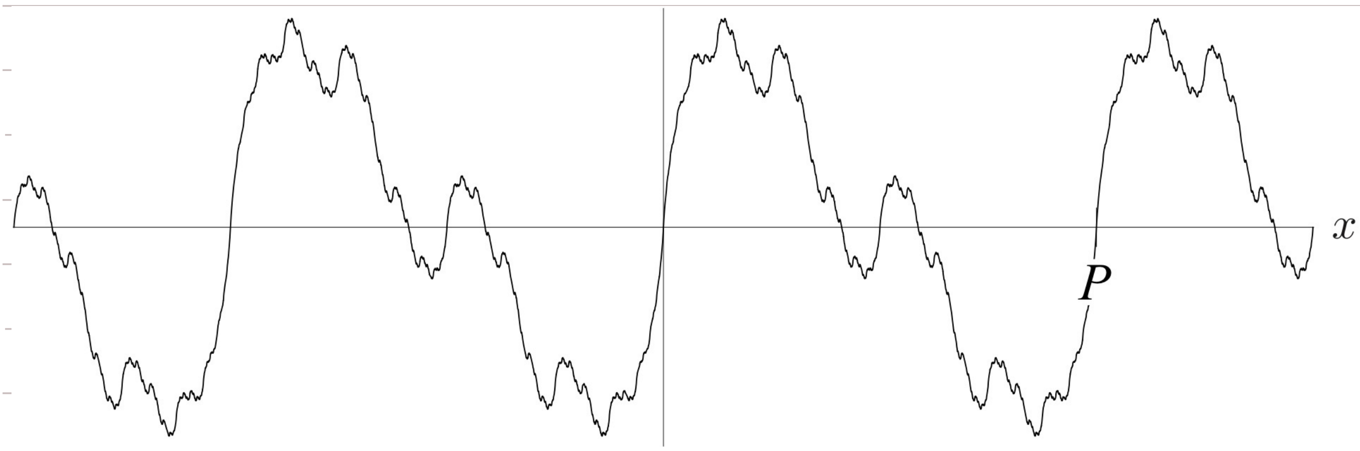
x
P

For simplicity, let's focus on 2π-periodic functions. For example, let $n \in \mathbb{Z}$. Then the function p_n defined by $p_n(x) = \cos(nx)$ is 2π-periodic, since

$$p_n(x+2\pi) = \cos(n(x+2\pi)) = \cos(nx + 2\pi n) = \cos(nx) = p_n(x).$$

Similarly, $q_n(x) = \sin(nx)$ and $e_n(x) = e^{inx} = \cos(nx) + i\sin(nx)$ define 2π-periodic functions.

Now let f be 2π-periodic. If Fourier was right, and f <u>has</u> a trigonometric series, then it stands to reason that the trigonometric <u>functions</u> in that series are 2π-periodic too. So, <u>maybe</u>, f has an expression of the form

$$f(x) = \sum_{n\in\mathbb{Z}} c_n(f)\, e_n(x) = \sum_{n\in\mathbb{Z}} c_n(f)\, e^{inx}$$

$$= \sum_{n\in\mathbb{Z}} c_n(f)\left[\cos(nx) + i\sin(nx)\right], \qquad (*)$$

for appropriate numbers $c_n(f) \in \mathbb{R}$ (or $\mathbb{C}$).

More on $(*)$ next time.