Chapter 5: Limits of functions.

Definition 5.1.1. Let $f : D \to \mathbb{R}$ and let c be an accumulation point of D. We say that a real number L is a limit of f at c, if for each $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - L| < \varepsilon$ whenever $x \in D$ and $0 < |x - c| < \delta$.

Theorem 5.1.8. Let $f : D \to \mathbb{R}$ and let c be an accumulation point of D. Then $\lim_{x\to c} f(x) = L$ iff for every sequence (s_n) in D that converges to c with $s_n \neq c$ for all n, the sequence $(f(s_n))$ converges to L.

Corollary 5.1.9. If $f : D \to \mathbb{R}$ and c is an accumulation point of D, then f can have only one limit at c.

Definition 5.2.1. Let $f : D \to \mathbb{R}$ and let $c \in D$. We say f **is continuous at** c if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - f(c)| < \varepsilon$ whenever $x \in D$ and $|x - c| < \delta$.

Theorem 5.2.2. Let $f : D \to \mathbb{R}$ and let $c \in D$. Then the following three conditions are equivalent: (a) f is continuous at c. (b) If (x_n) is any sequence in D such that (x_n) converges to c, then $\lim_{n\to\infty} f(x_n) = f(c)$. (c) For every neighborhood V of $f(c)$ there exists a neighborhood U of c such that $f(U \cap D) = V$. Furthermore, if c is an accumulation point of D, then the above are all equivalent to: (d) f has a limit at c and $\lim_{x\to c} f(x) = f(c)$.

Theorem 5.3.2. Let D be a compact subset of $\mathbb{R}$ and suppose that $f : D \to \mathbb{R}$ is continuous. Then $f(D)$ is compact.

Corollary 5.3.3. Let D be a compact subset of $\mathbb{R}$ and suppose that $f : D \to R$ is continuous. Then f assumes minimum and maximum values on D. That is, there exist points x_1 and x_2 in D such that $f(x_1) \leq f(x) \leq f(x_2)$ for all $x \in D$.

Chapter 6: Differentiation.

(It's assumed that you know the definition of the derivative, as well as the rules for differentiating sums of two functions, and constants times functions, as well as the product, quotient, and chain rules.)

Theorem 6.1.6. If $f : I \to \mathbb{R}$ is differentiable at a point $c \in I$, then f is continuous at c.

Theorem 6.2.1. If f is differentiable on an open interval (a, b) and if f assumes its maximum or minimum at a point $c \in (a, b)$, then $f'(c) = 0$.

Theorem 6.2.2. (Rolle's Theorem) Let f be a continuous function on $[a, b]$ that is differentiable on (a, b) and such that $f(a) = f(b)$. Then there exists at least one point c in (a, b) such that $f'(c) = 0$.

Theorem 6.2.3. (Mean Value Theorem) Let f be a continuous function on $[a, b]$ that is differentiable on (a, b). Then there exists at least one point $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Theorem 6.3.2. (l'Hôpital's Rule) Let f and g be continuous on $[a, b]$ and differentiable on (a, b). Suppose that $c \in [a, b]$ and that $f(c) = g(c) = 0$. Suppose also that $g'(x) \neq 0$ for $x \in U$, where U is the intersection of (a, b) and some deleted neighborhood of c. If $\lim_{x\to c}(f'(x)/g'(x)) = L$ then $\lim_{x\to c}(f(x)/g(x)) = L$.

Chapter 7: Integration.

(It's assumed that you know the definition of a partition P, of upper and lower sums $L(f, P)$ and $U(f, P)$ corresponding to a bounded function f and a partition P on $[a, b]$, of a refinement of a partition, of upper and lower integrals $U(f)$ and $L(f)$, and of the integral $\int_a^b f = \int_a^b f(x)\,dx$.)

Theorem 7.1.4. Let f be a bounded function on $[a, b]$. If P and Q are partitions of $[a, b]$ and Q is a refinement of P, then

$$L(f, P) \leq L(f, Q) \leq U(f, Q) \leq U(f, P).$$

Theorem 7.1.6. Let f be a bounded function on $[a, b]$. Then $L(f) \leq U(f)$.

Theorem 7.1.9. Let f be a bounded function on $[a, b]$. Then f is integrable iff for each $\varepsilon > 0$ there exists a partition P of $[a, b]$ such that $U(f, P) - L(f, P) < \varepsilon$.

Theorem 7.2.1. If f is monotone on $[a, b]$, then f is integrable on $[a, b]$.

Theorem 7.2.2. If f is continuous on $[a, b]$, then f is integrable on $[a, b]$.

Corollary 7.2.8. Let f be integrable on $[a, b]$. Then $|f|$ is integrable on $[a, b]$ and $\left|\int_a^b f\right| \leq \int_a^b |f|$.

Theorem 7.3.1. (The Fundamental Theorem of Calculus I) Let f be integrable on $[a, b]$. For each $x \in [a, b]$, let $F(x) = \int_a^x f(t)\, dt$. If f is continuous at $c \in [a, b]$, then F is differentiable at c and $F'(c) = f(c)$.

Theorem 7.3.2. (The Fundamental Theorem of Calculus II) If f is differentiable on $[a, b]$ and f' is integrable on $[a, b]$, then $\int_a^b f' = f(b) - f(a)$.

Chapter 8. Infinite series.

(It's assumed that you know the definition of an infinite series $\sum a_n$ as the limit of the sequence of partial sums of the sequence (a_n), and that you know what it means for a series to be convergent, conditionally convergent, absolutely convergent, or divergent.)

Theorem 8.1.5. (nth Term Test or Divergence Test) If $\sum a_n$ is a convergent series, then $\lim a_n = 0$.

Example 8.1.7. (Geometric Series Test) The series $\sum_{n=0}^{\infty} r^n$ converges to $1/(1-r)$ if $|r| < 1$ and diverges if $|r| \geq 1$.

Theorem 8.2.1. (Comparison Test) Let $\sum a_n$ and $\sum b_n$ be infinite series of nonnegative terms. Then (a) If $\sum a_n$ converges and $0 \leq b_n \leq a_n$ for all n, then $\sum b_n$ converges. (b) If $\sum a_n$ diverges and $0 \leq a_n \leq b_n$ for all n, then $\sum b_n$ diverges.

Theorem 8.2.7. (Ratio Test) Let $\sum a_n$ be a series of nonzero terms. (a) If $\lim |a_{n+1}/a_n| < 1$, then the series converges absolutely. (b) If $\lim |a_{n+1}/a_n| > 1$, then the series diverges. (c) If $\lim |a_{n+1}/a_n| = 1$, then the test gives no information about convergence or divergence.

Theorem 8.2.13. (Integral Test) Let f be a continuous function defined on $[1, \infty)$, and suppose that f is positive and decreasing. Then the series $\sum f(n)$ converges iff

$$\lim_{k \to \infty} \int_1^k f(x)\, dx$$

exists as a real number.

Theorem 8.2.16. (Alternating Series Test) If (a_n) is a decreasing sequence of positive numbers and $\lim a_n = 0$, then the series $\sum (-1)^n a_n$ converges.

Fourier series.

Basic Definitions. If $z = x + iy$ is a complex number, with $x, y \in \mathbb{R}$ and $i^2 = -1$, then we call x the **real part** of z, denoted Re z; we call y the **imaginary part** of z, denoted Im z; we call $x - iy$ the **complex conjugate** of z, denoted $\overline{z}$; we call $\sqrt{x^2 + y^2}$ the **modulus** of z, denoted $|z|$; we call the angle (in $(-\pi, \pi]$) that z makes with the positive x axis the **argument** of z, denoted Arg z. And for $z \neq 0 + i0 = 0$, we denote by z^{-1} or $1/z$ the unique complex number such that $zz^{-1} = z^{-1}z = 1$.

Complex exponentials. We write e^{ix} for the complex number $\cos x + i \sin x$. We have: (a) $|e^{ix}| = 1$. (b) $1/e^{ix} = e^{-ix}$. (c) $e^{ix_1} e^{ix_2} = e^{i(x_1 + x_2)}$. (d) $\left(e^{ix}\right)^n = e^{inx}$ for $n \in \mathbb{Z}$. (e) $e^{in\pi} = (-1)^n$ for $n \in \mathbb{Z}$.

Fourier series. If f is 2π-periodic (meaning $f(x + 2\pi) = f(x)$ for all x), f is continuous, and f' is piecewise continuous, then for all $x \in \mathbb{R}$,

$$f(x) = \sum_{n \in \mathbb{Z}} c_n(f) e^{inx} \quad \text{where} \quad c_n(f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx}\, dx.$$