I. Some definitions concerning sets and functions.

(i) Let A, B be sets. We define $A \cup B = \{x : x \in A \text{ or } x \in B\}$, $A \cap B = \{x : x \in A \text{ and } x \in B\}$, $A \backslash B = \{x \in A \colon x \notin B\}$.

(ii) Let A and B be sets. A **function** from A to B is a nonempty relation $f \subseteq A \times B$ that satisfies the following two conditions: **(a)** Existence: for all $a \in A, \exists b \in B \colon (a, b) \in f$. **(b)** Uniqueness: If $(a, b) \in f$ and $(a, c) \in f$, then $b = c$.
If f is a function from A to B and $(a, b) \in f$, then we write $f(a) = b$.

(iii) If f is a function from A to B, then we write $f : A \to B$, and A is called the **domain** of f, B is called the **codomain** of f, and the set $\{b \in B : \exists a \in A : f(a) = b\}$ is called the **range** of f.

(iv) If $f : A \to B$ and $S \subseteq B$, then we define $f^{-1}(S)$ to be the set $\{a \in A : f(a) \in S\}$.

(v) A function $f \colon A \to B$ is called **injective** (or **one-to-one**) if, for all a and a' in A, $f(a) = f(a')$ implies that $a = a'$.

(vi) A function $f \colon A \to B$ is called **surjective** (or **onto**) if, $\forall b \in B, \exists a \in A : f(a) = b$ (so that B equals the range of f).

(vii) A function $f \colon A \to B$ is called **bijective** if it is both injective and surjective.

II. Axioms of the real numbers.

“Addition” Axioms

A1. $\forall x, y \in \mathbb{R}$, $x + y \in \mathbb{R}$ and, if $x = w$ and $y = z$, then $x + y = w + z$.

A2. $\forall x, y \in \mathbb{R}$, $x + y = y + x$.

A3. $\forall x, y, z \in \mathbb{R}$, $x + (y + z) = (x + y) + z$.

A4. There is a unique real number 0 such that $x + 0 = x$, for all $x \in \mathbb{R}$.

A5. For each $x \in \mathbb{R}$, there is a unique real number $-x$ such that $x + (-x) = 0$.

“Multiplication” Axioms

M1. $\forall x, y \in \mathbb{R}$, $x \cdot y \in \mathbb{R}$ and, if $x = w$ and $y = z$, then $x \cdot y = w \cdot z$.

M2. $\forall x, y \in \mathbb{R}$, $x \cdot y = y \cdot x$.

M3. $\forall x, y, z \in \mathbb{R}$, $x \cdot (y \cdot z) = (x \cdot y) \cdot z$.

M4. There is a unique real number 1 such that $1 \neq 0$ and $x \cdot 1 = x$, for all $x \in \mathbb{R}$.

M5. For each $x \in \mathbb{R}$ with $x \neq 0$, there is a unique real number $1/x$ such that $x \cdot (1/x) = 1$. We also write x^{-1} or $\dfrac{1}{x}$ in place of $1/x$.

DL. For all $x, y, z \in \mathbb{R}$, $x \cdot (y + z) = x \cdot y + x \cdot z$.

“Order” Axioms

O1. $\forall x, y \in \mathbb{R}$, exactly one of the relations $x = y$, $x > y$, or $x < y$ holds **(trichotomy law)**.

O2. $\forall x, y, z \in \mathbb{R}$, if $x < y$ and $y < z$, then $x < z$.

O3. $\forall x, y, z \in \mathbb{R}$, if $x < y$, then $x + z < y + z$.

O4. $\forall x, y, z \in \mathbb{R}$, if $x < y$ and $z > 0$, then $x \cdot z < y \cdot z$.

“Completeness” Axiom

Every nonempty subset S of $\mathbb{R}$ that is bounded above has a least upper bound. That is, $\sup S$ exists as a real number.

III. The definition of a limit. Let $s \in \mathbb{R}$ and let (s_n) a sequence of real numbers. We say that **the sequence (s_n) converges to s**, and write

$$\lim_{n\to\infty} s_n = s, \qquad \text{or} \qquad \lim s_n = s, \qquad \text{or} \qquad s_n \to s,$$

if

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} : n \geq N \Rightarrow |s_n - s| < \varepsilon.$$

IV. Some definitions concerning the topology of $\mathbb{R}$.

(i) Let $S \subseteq \mathbb{R}$. **(a)** The **supremum** of S, denoted $\sup S$, is the least upper bound for S (provided an upper bound for S exists). **(a)** The **infemum** of S, denoted $\inf S$, is the greatest lower bound for S (provided a lower bound for S exists).

(ii) A **neighborhood** of a point $x \in \mathbb{R}$ is a set $N(x, \varepsilon) = (x - \varepsilon, x + \varepsilon)$, for some $\varepsilon > 0$.

(iii) A deleted neighborhood of a point $x \in \mathbb{R}$ is a set $N^*(x, \varepsilon) = (x - \varepsilon, x) \cup (x, x + \varepsilon)$, for some $\varepsilon > 0$.

(iv) An interior point of set $S \subseteq \mathbb{R}$ is a point $x \in \mathbb{R}$ such that, for some $\varepsilon > 0$, $N(x, \varepsilon) \subseteq S$. The set of all interior points of S is denoted $\operatorname{int} S$.

(v) A boundary point of set $S \subseteq \mathbb{R}$ is a point $x \in \mathbb{R}$ such that, for all $\varepsilon > 0$, $N(x, \varepsilon) \cap S \neq \emptyset$ and $N(x, \varepsilon) \cap \mathbb{R} \backslash S \neq \emptyset$. The set of all boundary points of S is denoted $\operatorname{bd} S$.

(vi) A set $S \subset \mathbb{R}$ is **closed** if $\operatorname{bd} S \subseteq S$. A set $S \subseteq \mathbb{R}$ is **open** if $\mathbb{R} \backslash S$ is closed.

(vii) An accumulation point of set $S \subseteq \mathbb{R}$ is a point $x \in \mathbb{R}$ such that, for every $\varepsilon > 0$, $N^*(x, \varepsilon) \cap S \neq \emptyset$. The set of all accumulation points of S is denoted S'.

(viii) An isolated point of set $S \subseteq \mathbb{R}$ is a point $x \in \mathbb{R}$ such that $x \in S$ but $x \notin S'$. The set of all isolated points of S simply the set $S \backslash S'$.

(ix) A set $S \subseteq \mathbb{R}$ is called **compact** if every open cover of S (that is, every collection $\{T_\alpha \colon \alpha \in \mathcal{A}\}$ of open sets T_α whose union contains S) has a finite subcover (meaning a collection of finitely many of the T_α's such that the union of these finitely many T_α's contains S).

(x) The **closure** $\operatorname{cl} S$ of a set $S \subseteq \mathbb{R}$ is defined by $\operatorname{cl} S = S \cup S'$.

V. Some theorems that you may use without proof (but you must cite the appropriate theorem at any point where it is needed).

(i) Theorem 3.1.2: The Principle of Mathematical Induction. Let A_n be a statement regarding a natural number n. Suppose that **(a)** A_1 is true, and **(b)** A_k implies A_{k+1}, for all $k \in \mathbb{N}$. Then A_n is true for all integers n.

(ii) Theorem 3.3.9: The Archimedean Property of $\mathbb{R}$. The set N of natural numbers is unbounded above in $\mathbb{R}$.

(iii) **Theorem 3.3.10:** Each of the following is equivalent to the Archimedean Property. **(a)** For each $z \in \mathbb{R}$, there exists an $n \in \mathbb{N}$ such that $n > z$. **(b)** For each $x > 0$ and for each $y \in \mathbb{R}$, there exists an $n \in \mathbb{N}$ such that $nx > y$. **(c)** For each $x > 0$, there exists an $n \in N$ such that $0 < 1/n < x$.

(iv) Theorem 3.3.13: The Density of $\mathbb{Q}$ in $\mathbb{R}$. If x and y are real numbers with $x < y$, then there exists a rational number r with $x < y < r$.

(v) Theorem 3.4.7. Let S be a subset of $\mathbb{R}$. **(a)** S is open iff $S = \operatorname{int} S$. **(b)** S is closed iff its complement $\mathbb{R} \backslash S$ is open.

(vi) Theorem 3.4.10 and Corollary 3.4.11. **(a)** The union of any collection of open sets is open. **(b)** The intersection of any finite collection of open sets is open. **(c)** The intersection of any collection of closed sets is closed. **(d)** The union of any finite collection of closed sets is closed.

(viii) Theorem 3.4.17. Let S be a subset of $\mathbb{R}$. **(a)** S is closed iff $S' \subseteq S$. **(b)** $\operatorname{cl} S$ is closed. **(c)** S is closed iff $S = \operatorname{cl} S$. **(d)** $\operatorname{cl} S = S \cup \operatorname{bd} S$.

(ix) Theorem 3.5.5 (Heine-Borel): A subset S of $\mathbb{R}$ is compact iff S is closed and bounded.

(x) Theorem 3.5.6 (Bolzano-Weierstrass): If $S \subseteq \mathbb{R}$ is bounded and contains infinitely many points, then there is at least one point in $\mathbb{R}$ such that $x \in S'$ (that is, such that x is an accumulation point of S).