Please fill in the indicated blanks and complete the indicated exercises.

1: Ternary expansions of real numbers in $[0, 1]$. By analogy with the idea of a *decimal* expansion, where any $x \in [0, 1]$ can be written using a string of digits between 0 and 9, it's not too hard to show that every real number $x \in [0, 1]$ has a *ternary* expansion, meaning an expression of the form

$$x = \frac{x_1}{3^1} + \frac{x_2}{3^2} + \frac{x_3}{3^3} + \frac{x_4}{3^4} + \frac{x_5}{3^5} + \frac{x_6}{3^6} + \cdots = \sum_{n=1}^{\infty} \frac{x_n}{3^n},$$

where, for each $n \in \mathbb{N}$, x_n is an integer satisfying $0 \le x_n \le 2$.

For example (fill in all missing numerators):

$$\frac{1}{3} = \frac{1}{3^1} + \frac{0}{3^2} + \frac{0}{3^3} + \frac{0}{3^4} + \frac{0}{3^5} + \frac{0}{3^6} + \cdots, \qquad (\mathrm{OT}_1)$$

$$\frac{\pi}{10} = \frac{0}{3^1} + \frac{2}{3^2} + \frac{2}{3^3} + \frac{1}{3^4} + \frac{1}{3^5} + \frac{1}{3^6} + \cdots, \qquad (\mathrm{OT}_2)$$

$$\frac{4}{27} = \frac{1}{9} + \frac{1}{27} = \frac{0}{3^1} + \frac{1}{3^2} + \frac{1}{3^3} + \frac{0}{3^4} + \frac{0}{3^5} + \frac{0}{3^6} + \cdots, \qquad (\mathrm{OT}_3)$$

$$1 = \frac{2}{3^1} + \frac{2}{3^2} + \frac{2}{3^3} + \frac{2}{3^4} + \frac{2}{3^5} + \frac{2}{3^6} + \cdots. \qquad (\mathrm{OT}_4)$$

Exercise 1. Use the geometric series formula

$$a \sum_{n=1}^{\infty} r^n = \frac{ar}{1-r} \quad (a \in \mathbb{R},\ |r| < 1) \qquad (\mathrm{GS})$$

to prove equation (OT_4).

$$\frac{2}{3^1} + \frac{2}{3^2} + \frac{2}{3^3} + \frac{2}{3^4} + \frac{2}{3^5} + \frac{2}{3^6} + \cdots$$

$$= 2\sum_{n=1}^{\infty}\left(\frac{1}{3}\right)^n = 2 \cdot \frac{1/3}{1 - 1/3} = 2 \cdot \frac{1}{3-1} = 2 \cdot \frac{1}{2} = 1.$$

To distinguish between decimal and ternary expansions, we might denote the latter by a subscript of "three." For example, by (OT_1), (OT_2), (OT_3), and (OT_4), we have $1/3 = 0.1_{\text{three}}$, $\pi/10 = 0.022111\ldots_{\text{three}}$, $1/27 =$ ___0.011___$_{\text{three}}$, $1 = 0.\overline{2}_{\text{three}}$. Here, the bar on top of the 2 means "repeat this forever."

2. Ternary expansions and the Cantor set. The Cantor set C is, by definition, the set of all real numbers in $[0, 1]$ that have a ternary expansion involving **only** 0's and 2's. That is, $x \in C$ if and only if we have an expression

$$x = \frac{x_1}{3^1} + \frac{x_2}{3^2} + \frac{x_3}{3^3} + \frac{x_4}{3^4} + \frac{x_5}{3^5} + \frac{x_6}{3^6} + \cdots$$

where $x_n = 0$ or $x_n = 2$ for each $n \in \mathbb{N}$.

Exercise 2. (a) Show that 1/4 has ternary expansion $1/4 = 0.\overline{02}_{\text{three}}$. Hint: by definition,

$$0.\overline{02}_{\text{three}} = 0.020202\ldots_{\text{three}} = \frac{0}{3^1} + \frac{2}{3^2} + \frac{0}{3^3} + \frac{2}{3^4} + \frac{0}{3^5} + \frac{2}{3^6} + \cdots = \frac{2}{9} + \frac{2}{9^2} + \frac{2}{9^3} + \cdots.$$

Now use the geometric series formula (GS) above.

By the hint and by (GS),

$$0.\overline{02}_{\text{three}} = \frac{2}{9} + \frac{2}{9^2} + \frac{2}{9^3} + \cdots = 2\sum_{n=1}^{\infty}\left(\frac{1}{9}\right)^n = 2 \cdot \frac{1/9}{1 - 1/9} = 2 \cdot \frac{1}{9-1} = \frac{2}{8} = \frac{1}{4}.$$

(b) Why can you conclude from part (a) that $1/4 \in C$?

By part (a), 1/4 has a ternary expansion involving only 0's and 1's, so by definition of the Cantor set C, $1/4 \in C$.

Note that, in defining the Cantor set, we used the phrase "*a* ternary expansion" instead of "*the* ternary expansion." This is because a number can have more than one! For example: we just saw (see, again, equation (OT_1)) that $1/3 = 0.1_{\text{three}}$. On the other hand:

Exercise 3. Use the geometric series formula (GS) above to show that

$$\frac{1}{3} = \frac{0}{3^1} + \frac{2}{3^2} + \frac{2}{3^3} + \frac{2}{3^4} + \frac{2}{3^5} + \frac{2}{3^6} + \cdots = 0.0\overline{2}_{\text{three}}.$$

Conclude that $1/3 \in C$.

By (GS),

$$\frac{0}{3^1} + \frac{2}{3^2} + \frac{2}{3^3} + \frac{2}{3^4} + \frac{2}{3^5} + \frac{2}{3^6} + \cdots = 2\sum_{n=2}^{\infty}\left(\frac{1}{3}\right)^n$$
$$= 2\left(\sum_{n=1}^{\infty}\left(\frac{1}{3}\right)^n - \frac{1}{3}\right) = 2\left(\frac{1/3}{1-1/3} - \frac{1}{3}\right) = 2\left(\frac{1}{2} - \frac{1}{3}\right) = 2 \cdot \frac{1}{6} = \frac{1}{3}.$$

So 1/3 has a ternary expansion involving only 0's and 2's, so $1/3 \in C$.

NOTE: Since C consists of all numbers that have a ternary expansion involving 0's and ___2___ 's only, we can get C by *removing*, from $[0,1]$, any number all of whose ternary expansions contain the integer ___1___ . Since $0 < 1 < 2$, it's not hard to see that: C is what you get by successively removing "open middle thirds" from the interval $[0,1]$. To be precise: we begin with the interval $[0,1]$, and remove the open interval $(\frac{1}{3}, \frac{2}{3})$ in the middle. Then, from what remains, we remove the "open middle thirds" that are left; that is, we remove $(\frac{1}{9}, \frac{2}{9})$ and $(\frac{7}{9}, \frac{8}{9})$. Then, from what remains, we again remove the open middle thirds. That is, we remove (___$\frac{1}{27}$___ , $\frac{2}{27}$), $(\frac{7}{27}, \frac{8}{27})$, $(\frac{19}{27}, \frac{20}{27})$, and ($\frac{25}{27}$, ___$\frac{26}{27}$___). And so on ad infinitum. Then C is, by definition, what's left. Here's a picture:

Exercise 4. Describe (using formulas and/or words) infinitely many elements of C. Hint: think about the endpoints of the intervals removed.

The endpoint of any interval that is removed from $[0,1]$ to get C remains in C. Think about it this way: at any stage, you move inward, away from any existing endpoints, to select the middle third to remove. So the endpoints that are there at any stage remain for subsequent stages. And new endpoints are introduced at every stage, since you're deleting open intervals at every stage.

For example: after removing the interval $(\frac{1}{3}, \frac{2}{3})$, you're left with the endpoints $0, \frac{1}{3}, \frac{2}{3}$, and 1. At the next stage, you're removing the middle thirds of the intervals $[0, \frac{1}{3}]$ and $[\frac{2}{3}, 1]$ that have these endpoints, so you're not removing these endpoints. At the next stage, you're removing the middle third of all remaining intervals, so you're not removing the endpoints of any of these intervals.

It's clear that, in the end, there are infinitely many endpoints. So C has infinitely many elements.

So C has infinitely many elements. In fact, it's not hard to show that C is *uncountable*, meaning you can't write down all of its elements in a (finite or infinite) list $x_1, x_2, x_3, \ldots$. (In a sense, uncountable sets have "more elements than $\mathbb{N}$" and "as many elements as $\mathbb{R}$." So C has a lot of elements!) It can also be shown that C is closed and bounded, and therefore, C is ___compact___ .

The familiar examples of compact, uncountable sets are closed, bounded intervals $[a,b]$ (with $a < b$). The final exercise shows that C is very different from any such example! (See next page.)

Exercise 5. Show that C does not contain *any* interval $[a, b]$ with $a < b$. Hint: suppose $a, b \in C$ and $a < b$. Suppose the first ternary digit at which a and b differ is the Kth digit. Can you change this digit to get a number c such that $a < c < b$ but $c \notin C$? If so, then the interval $[a, b]$ can't be in C, so you're done.

(It might help to think of an example. E.g. the first ternary digit at which $a = 0.022020_{\text{three}}$ and $b = 0.022220_{\text{three}}$ differ is the fourth one. And we know that $a < b$ because the fourth digit of a is 0, but the fourth digit of b is 2. Change the fourth digit of a to get something between a and b, but not in C.)

It's instructive to look first at our example. Since $a = 0.022020_{\text{three}}$ has a 0 in its fourth ternary digit, and $b = 0.022220_{\text{three}}$ has a 2 there, we can insert a number c in between by giving c a 1 in its first digit. Like this:

$$\begin{aligned} a &= 0.022020_{\text{three}}, \\ c &= 0.022120_{\text{three}}, \\ b &= 0.022220_{\text{three}}. \end{aligned}$$

It doesn't even matter what happens with a, b, or c after the fourth digit. The point is that their first three digits agree, but the fourth digit of a is less that that of c, which is less than that of b. This is enough to assure that $a < c < b$. And since c has a 1 in it, $c \notin C$. So the interval $[a, b]$ cannot sit entirely in C.

In general: suppose a and b are in C, and the first ternary digit at which they differ is the Kth. Then a and b might look like this:

$$\begin{aligned} a &= a_1 a_2 a_3 \cdots a_K a_{K+1} a_{K+2} \cdots, \\ b &= a_1 a_2 a_3 \cdots b_K b_{K+1} b_{K+2} \cdots, \end{aligned}$$

where $a_K \neq b_K$, since the Kth digit is, by assumption, the first one where a and b differ. (Before the Kth digit, the digits of a and b agree, since we're assuming the Kth digit is the first one at which they disagree. Beyond the Kth digit, digits of a and b may agree or they may be not; it doesn't matter.) Now note that, since a and b are in C, a_K must equal 0 or 2, and b_K must equal 0 or 2. In fact, since we're assuming $a < b$, it MUST be the case that $a_K < b_K$. Since a_K and b_K are each equal to 0 or 2, it MUST be the case that $a_K = 0$ and $b_K = 2$.

So create a new number c whose digits agree with those of a and b, up through the $(K-1)$st digit, but such that c has a 1 in the Kth digit. It doesn't matter what happens after that.

$$\begin{aligned} a &= a_1 a_2 a_3 \cdots 0 a_{K+1} a_{K+2} \cdots, \\ c &= a_1 a_2 a_3 \cdots 1 c_{K+1} c_{K+2} \cdots, \\ b &= a_1 a_2 a_3 \cdots 2 b_{K+1} b_{K+2} \cdots. \end{aligned}$$

We have $a < c < b$, and $c \notin C$ (since c has a 1 in its ternary expansion). So $[a, b] \not\subseteq C$.