Cauchy sequences

Overview: a convergent sequence (s_n), as we know, is one whose terms get closer and closer to a "target," or limit, s. Here we look at *Cauchy* sequences, which are sequences whose terms get closer and closer to *each other*.

Definition 4.3.9. The sequence (s_n) is said to be a **Cauchy sequence** (or is said to satisfy the **Cauchy criterion**, or is simply said to be **Cauchy**) if, given $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that, if $m, n \geq N$, then

$$|s_n - s_m| < \varepsilon.$$

Example 1. Show that the sequence $(1/n)$ is Cauchy.

Solution (fill in the blanks). Let $\varepsilon > 0$. [Scratchwork: We want N large enough that, if $m, n \geq N$, then

$$\left|\frac{1}{n} - \frac{1}{m}\right| < \underline{\quad \varepsilon \quad}.$$

But

$$\left|\frac{1}{n} - \frac{1}{m}\right| \leq \frac{1}{n} + \frac{1}{m}, \tag{$*$}$$

by the ___triangle___ inequality. If n and m are both larger than $2/\varepsilon$, then the right hand side of $(*)$ is $< \varepsilon/2 + \varepsilon/2 =$ ___ε___, and we're done. So this is what we write.] Let N be any integer larger than ___$\varepsilon/2$___. Then

$$m, n \geq N \Rightarrow \left|\frac{1}{n} - \frac{1}{m}\right| \leq \frac{1}{n} + \frac{1}{m} \leq \frac{1}{N} + \underline{\quad \frac{1}{N} \quad} = \frac{2}{N} < \frac{2}{2/\varepsilon} = \underline{\quad \varepsilon \quad}.$$

So the sequence $(1/n)$ is ___Cauchy___ . □

If it seems like there's not much difference between being convergent and being Cauchy, that's because there isn't.

Theorem 4.3.12 (The Cauchy convergence criterion). The sequence (s_n) of real numbers is convergent iff it is Cauchy.

Proof (fill in the blanks). First we prove that convergent $\Rightarrow$ Cauchy: Suppose (s_n) is convergent; let $s = \lim s_n$. Let $\varepsilon > 0$. [Scratchwork: Note that

$$|s_n - s_m| = |(s_n - s) + (s - s_m)| \leq |s_n - s| + |s - s_m|, \tag{$**$}$$

by the ___triangle___ inequality. Since (s_n) is convergent, we can make each term on the right hand side of $(**)$ less than $\varepsilon/2$ if m and ___n___ are large enough. So this is what we write.] Since (s_n) is convergent, there is an $N \in$ $\underline{\quad \mathbb{N} \quad}$ such that, if $n \geq N$, then $|s_n - s| < \varepsilon/2$. But then, by $(**)$,

$$m, n \geq N \Rightarrow |s_n - s_m| \leq |s_n - s| + \underline{\quad |s - s_m| \quad} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \underline{\quad \varepsilon \quad}.$$

So (s_n) is Cauchy.

Proving that Cauchy $\Rightarrow$ convergent is just a little bit harder; we omit the proof. □

Sometimes, to show that a sequence is *divergent*, the best strategy is to show that the sequence is *not Cauchy*, and then to use the above theorem.

Example. Show that the "harmonic series"

$$\sum_{k=1}^{\infty} \frac{1}{k}$$

does not converge.

Solution (fill in the blanks). Let s_n be the nth partial sum of this series; that is,

$$s_n = \sum_{k=1}^{n} \frac{1}{k}.$$

We wish to show that the sequence (s_n) does not $\underline{\quad \text{converge} \quad}$. To do so, we will show that this sequence is not $\underline{\quad \text{Cauchy} \quad}$, and then use Theorem 4.3.12.

To show that (s_n) is not Cauchy, let n be any positive integer. Note that

$$s_{2n} - s_n = \sum_{k=1}^{2n} \frac{1}{k} - \sum_{k=1}^{n} \frac{1}{k} = \sum_{k=n+1}^{2n} \frac{1}{k} = \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \underline{\quad \frac{1}{2n} \quad}. \quad (***)$$

On the right hand side of $(***)$, there are n terms, and the smallest of these terms equals $1/(2n)$. So the right hand side of $(***)$ is larger than

$$n \cdot \frac{1}{2n} = \frac{1}{2}.$$

So, by $(***)$, $s_{2n} - s_n$ is also larger than $\underline{\quad 1/2 \quad}$. So $|s_m - s_n|$ will always be larger than ε, as long as $m = \underline{\quad 2n \quad}$ and $\varepsilon = \underline{\quad 1/2 \quad}$. So the sequence (s_n) is not Cauchy, and therefore, by Theorem 4.3.12, does not $\underline{\quad \text{converge} \quad}$.