Take-Home Midterm Exam: SOLUTIONS

1. Recall that, by definition, the closure $\operatorname{cl} S$ of a set $S \subset \mathbb{R}$ is defined by $\operatorname{cl} S = S \cup S'$, where S' is the set of accumulation points of S.

   Prove Theorem 3.4.17(c): $\operatorname{cl} S = S \cup \operatorname{bd} S$. Do so by showing that (a) $\operatorname{cl}(S) \subseteq S \cup \operatorname{bd} S$ and (b) $S \cup \operatorname{bd} S \subseteq \operatorname{cl}(S)$. Use only the definitions of $\operatorname{cl} S$, S', and $\operatorname{bd} S$, in terms of neighborhoods, deleted neighborhoods, and so on. (Of course, you can use the definition of the union of two sets as well.)

   Proof. (a) We wish to show that $S \cup S' \subseteq S \cup \operatorname{bd} S$. So let $x \in S \cup S'$. Then $x \in S$ or $x \in S'$, by definition of union. If $x \in S$, then we're done, since then certainly $x \in S \cup \operatorname{bd} S$, by definition of union. If not, then $x \in S'$, again by definition of union. In this case, given $\varepsilon > 0$, we know that $N^*(x, \varepsilon)$ intersects S, by definition of accumulation point. Let y be a point in this intersection. Then $y \in N^*(x, \varepsilon)$, and since $N^*(x, \varepsilon) \subseteq N(x, \varepsilon)$, we also have $y \in N(x, \varepsilon)$. Since $y \in S$, then, we have $y \in N(x, \varepsilon) \cap S$. That is, $N(x, \varepsilon)$ intersects S. Moreover, $N(x, \varepsilon)$ also intersects $\mathbb{R} \backslash S$, since we're assuming $x \notin S$, so $x \in N(x, \varepsilon) \cap \mathbb{R} \backslash S$. So $N(x, \varepsilon)$ intersects both S and $\mathbb{R} \backslash S$. This is true for arbitrary $\varepsilon > 0$, so x is a boundary point of S, by definition of boundary point. So $x \in \operatorname{bd} S$, and consequently $x \in S \cup \operatorname{bd} S$, by definition of union. So $S \cup S' \subseteq S \cup \operatorname{bd} S$.

   (a) We wish to show that $S \cup \operatorname{bd} S \subseteq S \cup S'$. So let $x \in S \cup \operatorname{bd} S$. Then $x \in S$ or $x \in \operatorname{bd} S$, by definition of union. If $x \in S$, then we're done, since then certainly $x \in S \cup S'$, by definition of union. If not, then $x \in \operatorname{bd} S$, again by definition of union. In this case, given $\varepsilon > 0$, we know that $N(x, \varepsilon)$ intersects both S and $\mathbb{R} \backslash S$, by definition of boundary point. Let y be a point in $N(x, \varepsilon) \cap S$. Then $y \in S$, so $y \neq x$, since we're assuming $x \notin S$. So $y \in N(x, \varepsilon) \backslash \{x\} = N^*(x, \varepsilon)$. Since, again, $y \in S$, we have $y \in N^*(x, \varepsilon) \cap S$. So $N^*(x, \varepsilon) \cap S \neq \emptyset$. This is true for arbitrary $\varepsilon > 0$, so x is an accumulation point of S, by definition of accumulation point. So $x \in S'$, and consequently $x \in S \cup S'$, by definition of union. So $S \cup \operatorname{bd} S \subseteq S \cup S'$.

   Since $S \cup \operatorname{bd} S \subseteq S \cup S'$ and $S \cup S' \subseteq S \cup \operatorname{bd} S$, we have $S \cup \operatorname{bd} S = S \cup S' = \operatorname{cl} S$, the last equality by definition of $\operatorname{cl} S$. □

2. For $n \in N$, define

$$s_n = 3 - \frac{4}{n^3}.$$

(a) Show that (s_n) is a Cauchy sequence, using only Definition 4.3.9 from your text. That is: don't use any limit laws like "the limit of a sum is the sum of the corresponding limits," and don't use any results like Lemma 4.3.10 or Theorem 4.3.12 that give other criteria for a sequence to be Cauchy.

Solution. Let $\varepsilon > 0$. Let N be any integer greater than $2/\sqrt[3]{\varepsilon}$. Then for $m, n \geq N$, we have

$$\begin{aligned} |s_n - s_m| &= \left|3 - \frac{4}{n^3} - \left(3 - \frac{4}{m^3}\right)\right| = \left|\frac{4}{m^3} - \frac{4}{n^3}\right| \\ &\leq \left|\frac{4}{m^3}\right| + \left|\frac{4}{n^3}\right| = \frac{4}{m^3} + \frac{4}{n^3} < \frac{4}{(2/\sqrt[3]{\varepsilon})^3} + \frac{4}{(2/\sqrt[3]{\varepsilon})^3} \\ &= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

So by definition of Cauchy sequence, (s_n) is Cauchy. □

(b) Show that the sequence (s_n) from part (a) of this problem converges to 3. Please use only Definition 4.1.2 from your text. That is: don't use any limit laws like "the limit of a sum is the sum of the corresponding limits," and don't use any results like Theorem 4.3.12 that give other criteria for a sequence to be convergent. (In particular, **do not** use the result of part (a) of this problem.)

Solution. Let $\varepsilon > 0$. Let N be any integer greater than $\sqrt[3]{4/\varepsilon}$. Then for $n \geq N$, we have

$$|s_n - 3| = \left|3 - \frac{4}{n^3} - 3\right| = \left|\frac{4}{n^3}\right| = \frac{4}{n^3} < \frac{4}{(\sqrt[3]{4/\varepsilon})^3} = \varepsilon.$$

So by definition of limit, $s_n \to 3$. □

3. Show *carefully* that $(0, 1)$ is **not** compact, by exhibiting an open cover $\mathcal{C}$ of $(0, 1)$ that has no finite subcover. Hint: Let $\mathcal{C} = \{(\frac{1}{n}, 1) : n \in \mathbb{N}\}$.

   Some notes: (a) Please complete this problem using the given strategy; do not use the Heine-Borel Theorem. (b) You may use without proof the fact that every finite set of integers has maximum element, and/or the fact that, given any finite set of integers (or real numbers), the elements of that set can be listed in increasing order. (c) Be careful about other assumptions. For example, if your argument requires the Archimedean property of $\mathbb{R}$ (see page 127 of our text), or any immediate consequences of the Archimedean property of $\mathbb{R}$ (for example, Theorem 3.3.10), then please state how and where you're using such results.

   Solution. Let $\mathcal{C} = \{(\frac{1}{n}, 1) : n \in \mathbb{N}\}$.

   Let $x \in (0, 1)$. By Theorem 3.3.10(c), There is an $n_0 \in \mathbb{N}$ such that $0 < \frac{1}{n_0} < x$. But then, since $x < 1$ by assumption, we have $x \in (\frac{1}{n_0}, 1)$. But then x is certainly in the union $\cup_{n\in\mathbb{N}}(\frac{1}{n}, 1)$. Moreover, each interval $(\frac{1}{n}, 1)$ is clearly open. So $\mathcal{C}$ is an open cover of $(0, 1)$.

   But no finite subcover of $\mathcal{C}$ contains $(0, 1)$, for the following reason. Let $\mathcal{B}$ be a finite subset of $\mathcal{C}$. Write

$$\mathcal{B} = \left\{ \left(\frac{1}{n_1}, 1\right), \left(\frac{1}{n_2}, 1\right), \dots, \left(\frac{1}{n_k}, 1\right) \right\},$$

   for some positive integer k. Let M be the largest of the integers $n_1, n_2, \dots, n_k$. Then $\frac{1}{M}$ is the smallest of the numbers $\frac{1}{n_1}, \frac{1}{n_2}, \dots, \frac{1}{n_k}$. Then $\frac{1}{M+1}$ is smaller than any of these numbers, so $\frac{1}{M+1}$ is not in any of the intervals in $\mathcal{B}$, so $\frac{1}{M+1}$ is not in the union of these intervals. So we have found a cover (namely, $\cup_{n\in\mathbb{N}}(\frac{1}{n}, 1)$) of $(0, 1)$ with no finite subcover. So $(0, 1)$ is not compact.

4. Let $f\colon \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = \frac{1}{2x^2+1}.$$

Show that

$$\lim_{x\to 2} f(x) = \frac{1}{9},$$

using only Definition 5.1.1 from your text. That is: don't use any limit laws like "the limit of a sum is the sum of the corresponding limits," and don't use any results like Theorem 5.1.8 that give other criteria for determining the limit of a function.

Solution. First, let's note that

$$|9(2x^2+1)| = 9(2x^2+1) \geq 9 > 1,$$

since x^2 is always ≥ 0.

Let $\varepsilon > 0$. Let $\delta = \min\{1, \varepsilon/10\}$. Then, if $0 < |x-2| < \delta$, we have

$$\begin{aligned}
\left|f(x) - \frac{1}{9}\right| &= \left|\frac{1}{2x^2+1} - \frac{1}{9}\right| = \left|\frac{9-(2x^2+1)}{9(2x^2+1)}\right| \\
&= \left|\frac{8-2x^2}{9(2x^2+1)}\right| < |8-2x^2| = |2(2-x)(2+x)| \\
&= 2|x-2|\,|x+2| = 2|x-2|\,|x-2+4| \leq 2|x-2|\,(|x-2|+4) \\
&< 2(\varepsilon/10)(1+4) = 10\cdot\varepsilon/10 = \varepsilon.
\end{aligned}$$

So

$$\lim_{x\to 2} f(x) = \frac{1}{9},$$

by definition of limit.

5. Use the Principle of Mathematical Induction (Theorem 3.1.2) to show that

$$\sum_{k=1}^{n} \frac{1}{4k^2-1} = \frac{1}{4\cdot 1^2-1} + \frac{1}{4\cdot 2^2-1} + \frac{1}{4\cdot 3^2-1} + \cdots + \frac{1}{4n^2-1}$$
$$= \frac{n}{2n+1}$$

for any positive integer n. (To be clear, you only need to show that the quantity on the far left equals the quantity on the far right. The stuff in the middle is there just to make it clear what the sum is that we're looking at.) Hint: $4n^3+8n^2+5n+1 = (n+1)(2n+1)^2$, and $4(n+1)^2-1 = (2(n+1)+1)(2(n+1)-1)$.

Solution. Let A_n be the statement

$$\sum_{j=1}^{n} \frac{1}{4j^2-1} = \frac{n}{2n+1}$$

Is A_1 true?

$$\frac{1}{4\cdot 1^2-1} = \frac{1}{3} = \frac{1}{2\cdot 1+1},$$

so A_1 is true.
Now assume A_k:

$$\sum_{j=1}^{k} \frac{1}{4j^2-1} = \frac{k}{2k+1}$$

Then

$$\begin{aligned}\sum_{j=1}^{k+1} \frac{1}{4j^2-1} &= \sum_{j=1}^{k} \frac{1}{4j^2-1} + \frac{1}{4(k+1)^2-1}\\ &= \frac{k}{2k+1} + \frac{1}{4(k+1)^2-1}\\ &= \frac{k(4(k+1)^2-1)+1\cdot(2k+1)}{(2k+1)(4(k+1)^2-1)} = \frac{4k^3+8k^2+5k+1}{(2k+1)(4(k+1)^2-1)}\\ &= \frac{(k+1)(2k+1)^2}{(2k+1)(2(k+1)+1)(2(k+1)-1)} = \frac{k+1}{2(k+1)+1},\end{aligned}$$

so A_{k+1} follows.
Since A_1 is true and $A_k \Rightarrow A_{k+1}$ for all $k \in \mathbb{N}$, we find by mathematical induction that A_n is true for all $n \in \mathbb{N}$.

6. Suppose the sequence (s_n) converges to some real number L. Show that the sequence (t_n) defined by

$$t_n = \frac{s_n}{n}$$

converges to zero. Please use only Definition 4.1.2 from your text. That is: don't use any limit laws like "the limit of a sum is the sum of the corresponding limits;" don't use the squeeze law; don't use any results like Theorem 4.3.12 that give other criteria for a sequence to be convergent. Hint:

$$|t_n - 0| = |t_n| = \left|\frac{s_n}{n}\right| = \left|\frac{s_n - L + L}{n}\right| \leq \left|\frac{s_n - L}{n}\right| + \left|\frac{L}{n}\right|.$$

Solution. Let $\varepsilon > 0$. Since $s_n \to L$, we know that there is an integer $M \in \mathbb{N}$ such that $n \geq M \Rightarrow |s_n - L| < \varepsilon/2$. Let N be any integer larger than $\max\{M, 2|L|/\varepsilon\}$. If $n \geq N$, then by the above hint,

$$\begin{aligned} |t_n - 0| &\leq \left|\frac{s_n - L}{n}\right| + \left|\frac{L}{n}\right| = \frac{|s_n - L|}{n} + \frac{|L|}{n} \\ &< \frac{\varepsilon/2}{n} + \frac{|L|}{(2|L|/\varepsilon)} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Strictly speaking, since we're dividing by $|L|$, these computations only hold for $L \neq 0$. But note that, if $L = 0$, then by the hint we have, for the same N,

$$n \geq N \Rightarrow |t_n - 0| \leq \left|\frac{s_n - L}{n}\right| + \left|\frac{L}{n}\right| = \left|\frac{s_n}{n}\right| = \frac{|s_n|}{n} < \frac{\varepsilon/2}{n} < \varepsilon.$$

So in any case we find that, by the definition of limit, $t_n \to 0$.