Take-Home Midterm Exam, due no later than 9:05 AM, March 17, 2023

You are to complete this midterm on your own, without assistance from any human or other resources, except for: our course text *Analysis with an Introduction to Proof*, lectures notes from this class, S-POP, any resources to which there is a *direct link* from our Canvas page (including the text *Book of Proof*, and any solutions I've posted to anything), yourself, and me (you are free to ask me questions, many of which I will probably not answer completely for you). You can also use paper and something to write with (or LaTeX). You almost certainly won't need a calculator, but if you think you do, ask me first. (These days, "calculators" can do all sorts of things, and I'll need to approve the device/app you're using first.)

Be *neat.* **You must complete, and attach to your exam, a copy the cover sheet at the end of this exam.**

Do not use any online or other resources if they were not mentioned above. E.g. Chegg is **bad,** and I will spot-check work against answers posted there and in similar places.

Here are the sections of the text to which this exam corresponds: Sections 2.3, 3.1–3.5, 4.1–4.3, and 5.1.

This course is about communicating mathematics as much as anything else, so please *communicate.* Make sure you state your assumptions, conclusions, and arguments along the way. (For example, in an induction problem, if you only prove A_1 and that A_k implies A_{k+1}, and don't tell me what your ultimate conclusion is, you will lose points. As another example, if you're doing a limit problem and you conclude with "$\ldots < \varepsilon$" or something like that, without stating what limit fact you've just shown, you'll lose points. If you say something like "$N(x, \varepsilon) \subseteq S$" without specifying whether you mean "for some $\varepsilon > 0$" or "for all $\varepsilon > 0$," you'll lose points. And so on.) And make sure you solve the problem in the manner specified. For example, don't use limit laws if it says to use only the definition of the limit.

You *do not* have to justify each time you use any of the axioms A1–A5, M1–M5, DL, O1–O4, **or** the Completeness Axiom, **or** the Archimedean Property, **or** the triangle inequality, **or** any of the immediate consequences like Theorems 3.2.2 and Theorem 3.2.10, unless you are **specifically asked** to do so. For example, you won't lose points if you write something like

$$|x^2 - 5| = |x-5|\,|x+5| = |x-5|\,|x-5+10| \le |x-5|\,\big(|x-5|+10\big)$$

without further justification, even though we're implicitly using lots of stuff (Axiom O4, Theorem 3.2.10(c) and (d), etc.) here.

If you *are* asked to be specific about such an argument, though, please do so, e.g. Problem 3 below.

Your exam must be turned in on or before the *beginning* of class on Friday, March 17. Electronic and hard copy submissions are both acceptable, as long as they are **neat**. Late exams will **not** be accepted.

If you would like a copy of the LaTeX file used to create this exam, send me an email.

Breathe. Good luck!

1. Recall that, by definition, the closure $\operatorname{cl} S$ of a set $S \subset \mathbb{R}$ is defined by $\operatorname{cl} S = S \cup S'$, where S' is the set of accumulation points of S.

   Prove Theorem 3.4.17(c): $\operatorname{cl} S = S \cup \operatorname{bd} S$. Do so by showing that (a) $\operatorname{cl}(S) \subseteq S \cup \operatorname{bd} S$ and (b) $S \cup \operatorname{bd} S \subseteq \operatorname{cl}(S)$. Use only the definitions of $\operatorname{cl} S$, S', and $\operatorname{bd} S$, in terms of neighborhoods, deleted neighborhoods, and so on. (Of course, you can use the definition of the union of two sets as well.)

   Some hints: (a) We wish to show that $S \cup S' \subseteq S \cup \operatorname{bd} S$. So let $x \in S \cup S'$. Then $x \in S$ or $x \in S'$ (explain why). If $x \in S$, then we're done. (Explain why.) If not, then $x \in S'$ (explain why). Then, given $\varepsilon > 0$, we know that $N^*(x, \varepsilon)$ intersects S (explain why). But then $N(x, \varepsilon)$ intersects S as well (explain why). Moreover, $N(x, \varepsilon)$ also intersects $\mathbb{R} \backslash S$ (explain why). So x is a boundary point of S (explain why). So $x \in \operatorname{bd} S$, and consequently $x \in S \cup \operatorname{bd} S$ (explain why). So $S \cup S' \subseteq S \cup \operatorname{bd} S$.

   The other direction ($S \cup \operatorname{bd} S \subseteq \operatorname{cl}(S)$) is similar.

2. For $n \in N$, define
$$s_n = 3 - \frac{4}{n^3}.$$

   (a) Show that (s_n) is a Cauchy sequence, using only Definition 4.3.9 from your text. That is: don't use any limit laws like "the limit of a sum is the sum of the corresponding limits," and don't use any results like Lemma 4.3.10 or Theorem 4.3.12 that give other criteria for a sequence to be Cauchy.

   (b) Show that the sequence (s_n) from part (a) of this problem converges to 3. Please use only Definition 4.1.2 from your text. That is: don't use any limit laws like "the limit of a sum is the sum of the corresponding limits," and don't use any results like Theorem 4.3.12 that give other criteria for a sequence to be convergent. (In particular, **do not** use the result of part (a) of this problem.)

3. Show *carefully* that $(0, 1)$ is **not** compact, by exhibiting an open cover $\mathcal{C}$ of $(0, 1)$ that has no finite subcover. Hint: Let $\mathcal{C} = \{(\frac{1}{n}, 1) : n \in \mathbb{N}\}$.

   Some notes: (a) Please complete this problem using the given strategy; do not use the Heine-Borel Theorem. (b) You may use without proof the fact that every finite set of integers has maximum element, and/or the fact that, given any finite set of integers (or real numbers), the elements of that set can be listed in increasing order. (c) Be careful about other assumptions. For example, if your argument requires the Archimedean property of $\mathbb{R}$ (see page 127 of our text), or any immediate consequences of the Archimedean property of $\mathbb{R}$ (for example, Theorem 3.3.10), then please state how and where you're using such results.

4. Let $f\colon \mathbb{R} \to \mathbb{R}$ be given by
$$f(x) = \frac{1}{2x^2 + 1}.$$
Show that
$$\lim_{x \to 2} f(x) = \frac{1}{9},$$
using only Definition 5.1.1 from your text. That is: don't use any limit laws like "the limit of a sum is the sum of the corresponding limits," and don't use any results like Theorem 5.1.8 that give other criteria for determining the limit of a function.

   Some hints: (a) First, by getting a common denominator, show that $|f(x) - 1/9| \le |8 - 2x^2|$. (b) Now
$$|8 - 2x^2| = |2(4 - x^2)| = |2(2 - x)(2 + x)|. \tag{$*$}$$
(c) Explain carefully why the right hand side of $(*)$ is less than or equal to $2|2 - x|\big(|x - 2| + 4\big)$. (d) Let $\delta = \min\{1, \varepsilon/10\}$, and deduce the desired result.

   Note: it's OK if you use a different strategy, and a different δ, to get $|f(x) - 1/9| < \varepsilon$, but you do need to use Definition 5.1.1, and not use limit laws, etc.

5. Use the Principle of Mathematical Induction (Theorem 3.1.2) to show that
$$\sum_{k=1}^{n} \frac{1}{4k^2 - 1} = \frac{1}{4 \cdot 1^2 - 1} + \frac{1}{4 \cdot 2^2 - 1} + \frac{1}{4 \cdot 3^2 - 1} + \cdots + \frac{1}{4n^2 - 1}$$
$$= \frac{n}{2n + 1}$$
for any positive integer n. (To be clear, you only need to show that the quantity on the far left equals the quantity on the far right. The stuff in the middle is there just to make it clear what the sum is that we're looking at.) Hint: $4n^3 + 8n^2 + 5n + 1 = (n+1)(2n+1)^2$, and $4(n+1)^2 - 1 = (2(n+1) + 1)(2(n+1) - 1)$.

6. Suppose the sequence (s_n) converges to some real number L. Show that the sequence (t_n) defined by
$$t_n = \frac{s_n}{n}$$
converges to zero. Please use only Definition 4.1.2 from your text. That is: don't use any limit laws like "the limit of a sum is the sum of the corresponding limits;" don't use the squeeze law; don't use any results like Theorem 4.3.12 that give other criteria for a sequence to be convergent. Hint:
$$|t_n - 0| = |t_n| = \left|\frac{s_n}{n}\right| = \left|\frac{s_n - L + L}{n}\right| \le \left|\frac{s_n - L}{n}\right| + \left|\frac{L}{n}\right|.$$

(Those are all the problems. Don't forget the cover page below.)

MATH 3001-001: Analysis I

March 17, 2023

TAKE-HOME MIDTERM EXAM

I have neither given nor received unauthorized assistance on this exam.

Name:____

Signature:____

DO NOT WRITE IN THIS BOX!

Problem	Points	Score
1	17 pts	
2	18 pts	
3	16 pts	
4	17 pts	
5	15 pts	
6	17 pts	
TOTAL	100 pts	