MATH 3001: Analysis I

March 3, 2023

In-class Midterm Exam

I have neither given nor received unauthorized assistance on this exam.

Name: SOLUTIONS

Signature: ______________________

You must **show your work** on every problem of this exam.

If you get stuck on a problem, move on to the next one,
and then come back.

Breathe. GOOD LUCK!!

DO NOT WRITE IN THIS BOX!

Problem	Points	Score
1	18 pts	
2	13 pts	
3	30 pts	
4	13 pts	
5	16 pts	
6	10 pts	
TOTAL	100 pts	

1. (18 points; 6 points each)

Mark each statement as true or false. Justify each answer: if the answer is "True," supply a careful proof, using definitions and theorems from your fact sheet. If the answer is "False," supply an *explicit* counterexample, and explain why it's a counterexample.

(a) For any set $S \subseteq \mathbb{R}$, $\operatorname{bd} S \cap \operatorname{int} S = \emptyset$.

True. Let $S \subseteq \mathbb{R}$ and let $x \in \operatorname{int} S$. By definition of interior point, there exists $\varepsilon > 0$ such that $N(x, \varepsilon) \subseteq S$. This implies that $N(x, \varepsilon) \cap (\mathbb{R} \backslash S) = \emptyset$. But then x cannot be a boundary point of S, because every neigborhood of a boundary point of S interesects both S and $\mathbb{R} \backslash S$.

(b) Let A and B be sets. If $f : A \to B$ is a function and D is a nonempty subset of B, then $f^{-1}(D)$ is a nonempty subset of A.

False. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$. The set $D = [-2, -1]$ is a nonempty subset of $\mathbb{R}$, but $f^{-1}(D)$ is empty, since no real numbers map onto negative numbers.

(c) For any sets $S, T \subseteq \mathbb{R}$, $\operatorname{bd} S \cup \operatorname{bd} T = \operatorname{bd}(S \cup T)$.

False. If $S = [-2, -1]$ and $T = [-1, 1]$, then $S \cup T = [-2, 1]$, so $\operatorname{bd}(S \cup T) = \operatorname{bd}[-2, 1] = \{-2, 1\}$, but $\operatorname{bd} S \cup \operatorname{bd} T = \operatorname{bd}[-2, -1] \cup \operatorname{bd}[-1, 1] = \{-2, -1\} \cup \{-1, 1\} = \{-2, -1, 1\}$.

2. (13 points) Using only the definition of limit given on the fact sheet, show *carefully* that

$$\lim_{n\to\infty} \frac{4n+3}{n+3} = 4.$$

Let $\varepsilon > 0$. Suppose N is any positive integer larger than $9/\varepsilon$. Then if $n \geq N$, we have

$$\begin{aligned}\left|\frac{4n+3}{n+3} - 4\right| &= \left|\frac{4n+3}{n+3} - 4\frac{n+3}{n+3}\right| \\ &= \left|\frac{-9}{n+3}\right| = \frac{9}{n+3} < \frac{9}{n} < \frac{9}{9/\varepsilon} \\ &= \varepsilon.\end{aligned}$$

So

$$\lim_{n\to\infty} \frac{4n+3}{n+3} = 4.$$

3. (a) (4 points for each blank) Find the interior, boundary, accumulation points, isolated points, and closure of the set

$$A = [-5, 1) \cup \left\{ 2 + \frac{1}{n} \mid n \in \mathbb{N} \right\}.$$

You don't need to justify your answers.

int $A =$ $(-5, 1)$

bd $A =$ $\{-5, 1, 2\} \cup \{2 + \frac{1}{n} \mid n \in \mathbb{N}\}$

$A' =$ $[-5, 1] \cup \{2\}$

$A \backslash A' =$ $\{2 + \frac{1}{n} \mid n \in \mathbb{N}\}$

cl $A =$ $[-5, 1] \cup \{2\} \cup \{2 + \frac{1}{n} \mid n \in \mathbb{N}\}$

(b) (5 points) Is A open, closed, or neither? Please explain carefully, using definitions and/or theorems from your fact sheet. A is not closed, because it does not contian the boundary point 1 of A. Also A is not open, because it does not equal its interior. For example, $-5 \in A$ but $-5 \notin \text{int } A$.)

(c) (5 points) Is A compact? Please explain carefully, using definitions and/or theorems from your fact sheet. A is not closed so, by the Heine-Borel Theorem, it is not compact.

4. (13 points) Use the Principle of Mathematical Induction to show that

$$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

for all $n \in \mathbb{N}$.

Let A_n be the statement

$$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}.$$

Is A_1 true? $1^3 = 1 = \frac{1^2(1+1)^2}{4}$, so A_1 is true.
Now assume A_k:

$$\sum_{i=1}^{k} i^3 = \frac{k^2(k+1)^2}{4}.$$

Then

$$\begin{aligned}\sum_{i=1}^{k+1} i^3 &= \sum_{i=1}^{k} i^3 + (k+1)^3 \\ &= \frac{k^2(k+1)^2}{4} + (k+1)^3 \\ &= (k+1)^2\left(\frac{k^2}{4} + (k+1)\right) \\ &= (k+1)^2\left(\frac{k^2+4k+4}{4}\right) \\ &= \frac{(k+1)^2(k+2)^2}{4} = \frac{(k+1)^2((k+1)+1)^2}{4}.\end{aligned}$$

so A_{k+1} follows.
Since A_1 is true and $A_k \Rightarrow A_{k+1}$ for all $k \in \mathbb{N}$, we find by mathematical induction that A_n is true for all $n \in \mathbb{N}$.

5. Fill in in the blanks (2 point per blank; there are eight blanks.)

Theorem. The set $\mathbb{N}$ of natural numbers is not compact.

Proof. To show $\mathbb{N}$ is not compact, we need to find an open cover of $\mathbb{N}$ with no finite subcover. That is, we need to find a collection $\mathcal{C}$ of ____open____ sets such that $\mathbb{N}$ is contained in the union of the sets in $\mathcal{C}$, but $\mathbb{N}$ is *not* contained in the union of any finite number of sets in ____$\mathcal{C}$____ .

Let $\mathcal{C} = \{I_n \colon n \in \mathbb{N}\}$, where I_n is the open interval $I_n = (n - \frac{1}{2}, n + \frac{1}{2})$. Then each I_n is open (since I_n is an open interval), and certainly

$$\mathbb{N} \subseteq \cup_{n \in \mathbb{N}} I_n$$

because, if n is a positive integer, then n is in the interval ____I_n____ . So $\mathcal{C}$ is an open ____cover____ of $\mathbb{N}$.

To show that $\mathcal{C}$ has no finite subcover of $\mathbb{N}$, consider any *finite* set of intervals of the form $(n - \frac{1}{2}, n + \frac{1}{2})$. Let's say there are K intervals in this finite set. List them in increasing order: that is, list them as

$$\left(n_1 - \frac{1}{2}, n_1 + \frac{1}{2}\right), \left(n_2 - \frac{1}{2}, n_2 + \frac{1}{2}\right), \left(n_3 - \frac{1}{2}, n_2 + \frac{1}{2}\right), \ldots, \left(n_K - \frac{1}{2}, n_K + \frac{1}{2}\right), \qquad (*)$$

where $n_1 < n_2 < n_3 < \cdots < n_K$. (It's a fact that every finite set of integers can be written in increasing order; proof omitted.) Since the integer $n_K + 1$ is larger than $n_K + \frac{1}{2}$, we see that the integer ____$n_K + 1$____ is not in any of the intervals in $(*)$, and therefore, is not in the union of these intervals.

So we've shown that every finite subcover of the open cover $\mathcal{C}$ of $\mathbb{N}$ *fails* to cover ____$\mathbb{N}$____ . In other words, we've found an open cover of $\mathbb{N}$ with no finite ____subcover____ . So $\mathbb{N}$ is not ____compact____ . $\square$

6. Consider the real numbers $\mathbb{R}$, with the usual multiplication, denoted by $\cdot$, and with an "addition" operator @ defined by

$$x \,@\, y = \text{ the mean (average) of } x \text{ and } y \;= \frac{x+y}{2}.$$

(a) (5 points) Show that, with these definitions of addition @ and multiplication $\cdot$, Axiom DL (the distributive law) holds. That is, show that $x \cdot (y \,@\, z) = (x \cdot y) \,@\, (x \cdot z)\ \forall x, y, z \in \mathbb{R}$.

We have

$$x \cdot (y \,@\, z) = x \cdot \left(\frac{y+z}{2}\right) = \frac{x \cdot y + x \cdot z}{2},$$

by our definitions of $\cdot$ and @, and by the regular distrivutive law for the usual addition +. On the other hand, also by these definitions,

$$(x \cdot y) \,@\, (x \cdot z) = \frac{x \cdot y + x \cdot z}{2}.$$

So $x \cdot (y \,@\, z) = (x \cdot y) \,@\, (x \cdot z)\ \forall x, y, z \in \mathbb{R}$.

(b) (5 points) Use an explicit counterexample to show that, with this definition of addition @, Axiom A3 (the associative law for addition) **does not** hold.

The associative law for @, were it to be true, would say that $x \,@\, (y \,@\, z) = (x \,@\, y) \,@\, z$ $\forall x, y, z \in \mathbb{R}$. But this is not the case: take, for example, $x = 11$, $y = 3$, $z = 7$. Then

$$x \,@\, (y \,@\, z) = 11 \,@\, (3 \,@\, 7) = 11 \,@\, \left(\frac{3+7}{2}\right) = 11 \,@\, 5 = \frac{11+5}{2} = 8,$$

while

$$(x \,@\, y) \,@\, z = (11 \,@\, 3) \,@\, 7 = \left(\frac{11+3}{2}\right) @\, 7 = 7 \,@\, 7 = \frac{7+7}{2} = 7.$$

And $8 \neq 7$.