Assignment:

Section 8.1, pages 307–308: Exercises 1, 2, 4, 5bcdgi, 9.

Section 8.2, pages 316–319: Exercises 3abdhm, 4a, 5adi, 6.

Section 8.1:

1. Mark each statement True or False. Justify each answer.

(a) The symbol $\sum_{n=1}^{\infty} a_n$ is used to denote the sequence of partial sums of the sequence a_n. **Solution: True.** As it says on p. 302:

"As defined above, the symbol $\sum_{n=1}^{\infty} a_n$ is used in two ways: It is used to denote the sequence (s_n) of partial sums, and it is also used to denote the limit of the sequence of partial sums, provided that this limit exists. This dual usage should not cause confusion, since the context will make the intended meaning clear."

(b) The symbol $\sum_{n=1}^{\infty} a_n$ is used to denote the limit of the sequence of partials sums of the sequence a_n. **Solution: True.** See part (a) of this exercise, above.

2. Mark each statement True or False. Justify each answer.

(a) $\sum a_n$ converges iff $\lim a_n = 0$. **Solution: False.** For example, the harmonic series $\sum 1/n$ diverges, even though $\lim_{n\to\infty} a_n = 0$.

(b) The geometric series $\sum r^n$ converges iff $|r| < 1$. **Solution: True.** See Example 8.1.7.

4. Show that each series is divergent.

(a) $\sum(-1)^n$ **Solution:** $\lim_{n\to\infty}(-1)^n \neq 0$, so the series diverges by the nth term test (Theorem 8.1.5).

(b) $\sum \frac{n}{2n+1}$ **Solution:** $\lim_{n\to\infty} \frac{n}{2n+1} = \frac{1}{2} \neq 0$, so the series diverges by the nth term test (Theorem 8.1.5).

(c) $\sum \frac{n}{\sqrt{n^2+1}}$ **Solution:** $\lim_{n\to\infty} \frac{n}{\sqrt{n^2+1}} = 1 \neq 0$, so the series diverges by the nth term test (Theorem 8.1.5).

(d) $\sum \cos\frac{n\pi}{2}$ **Solution:** Note that $\cos\frac{n\pi}{2} = 0$ if n is odd, equals 1 if n is a multiple of 4, and equals -1 if n is a multiple of 2 but not a multiple of 4. So $\lim_{n\to\infty}\cos\frac{n\pi}{2} \neq 0$, so the series diverges by the nth term test (Theorem 8.1.5).

5. Find the sum of each series.

(b) $\sum_{n=3}^{\infty}\left(\frac{1}{2}\right)^n$ **Solution:**

$$\sum_{n=3}^{\infty}\left(\frac{1}{2}\right)^n = \sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n - \sum_{n=0}^{2}\left(\frac{1}{2}\right)^n = \frac{1}{1-1/2} - \left(1+\frac{1}{2}+\frac{1}{4}\right) = 2-\frac{7}{4} = \frac{1}{4}.$$

(c) $\sum_{n=0}^{\infty} 2\left(-\frac{1}{2}\right)^n$ **Solution:**

$$\sum_{n=0}^{\infty} 2\left(-\frac{1}{2}\right)^n = 2\sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^n = 2\frac{1}{1-(-1/2)} = 2\cdot\frac{2}{3} = \frac{4}{3}.$$

(d) $\sum_{n=1}^{\infty}\left(-\frac{3}{4}\right)^n$ **Solution:**

$$\sum_{n=1}^{\infty}\left(-\frac{3}{4}\right)^n = \sum_{n=0}^{\infty}\left(-\frac{3}{4}\right)^n - 1 = 2\frac{1}{1-(-3/4)} - 1 = \frac{4}{7} - 1 = -\frac{3}{7}.$$

(g) $\sum_{n=1}^{\infty}\frac{1}{(3n-2)(3n+1)}$ **Solution:**

$$\begin{aligned}
&\sum_{n=1}^{\infty}\frac{1}{(3n-2)(3n+1)} = \sum_{n=1}^{\infty}\frac{1}{3}\left[\frac{1}{3n-2}-\frac{1}{3n+1}\right]\\
&= \frac{1}{3}\lim_{k\to\infty}\left(\left[\frac{1}{1}-\frac{1}{4}\right]+\left[\frac{1}{4}-\frac{1}{7}\right]+\left[\frac{1}{7}-\frac{1}{10}\right]+\cdots+\left[\frac{1}{3k-2}-\frac{1}{3k+1}\right]\right)\\
&= \frac{1}{3}\lim_{k\to\infty}\left[1-\frac{1}{3k+1}\right] = \frac{1}{3}\left[1-\lim_{k\to\infty}\frac{1}{3k+1}\right] = \frac{1}{3}[1-0] = \frac{1}{3}.
\end{aligned}$$

(i) $\sum_{n=1}^{\infty}\frac{1}{n^2+3n+2}$ **Solution:**

$$\begin{aligned}
&\sum_{n=1}^{\infty}\frac{1}{n^2+3n+2} = \sum_{n=1}^{\infty}\frac{1}{(n+1)(n+2)} = \sum_{n=1}^{\infty}\left[\frac{1}{n+1}-\frac{1}{n+2}\right]\\
&= \lim_{k\to\infty}\left(\left[\frac{1}{2}-\frac{1}{3}\right]+\left[\frac{1}{3}-\frac{1}{4}\right]+\left[\frac{1}{4}-\frac{1}{5}\right]+\cdots+\left[\frac{1}{k+1}-\frac{1}{k+2}\right]\right)\\
&= \lim_{k\to\infty}\left[\frac{1}{2}-\frac{1}{k+2}\right] = \frac{1}{2}-\lim_{k\to\infty}\frac{1}{k+2} = \frac{1}{2}-0 = \frac{1}{2}.
\end{aligned}$$

9. Determine whether or not the series $\sum_{n=1}^{\infty} 1/(\sqrt{n+1}+\sqrt{n})$ converges. Justify your answer. **Solution:**

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{1}{\sqrt{n+1}+\sqrt{n}} &= \sum_{n=1}^{\infty}\frac{1}{\sqrt{n+1}+\sqrt{n}}\cdot\frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n+1}-\sqrt{n}}\\
&= \sum_{n=1}^{\infty}\frac{\sqrt{n+1}-\sqrt{n}}{n+1-n} = \sum_{n=1}^{\infty}(\sqrt{n+1}-\sqrt{n})\\
&= \lim_{k\to\infty}\left([\sqrt{2}-\sqrt{1}]+[\sqrt{3}-\sqrt{2}]+[\sqrt{4}-\sqrt{3}]+\cdots+[\sqrt{k+1}-\sqrt{k}]\right)\\
&= \lim_{k\to\infty}\left(-\sqrt{1}+\sqrt{k+1}\right) = \infty.
\end{aligned}$$

The limit of the partial sums does not exist as a finite number, so the series diverges.

Section 8.2:

3. Determine whether each series converges or diverges. Justify your answer.

(a) $\sum \frac{n^3}{3^n}$ **Solution:** Converges by the ratio test, since

$$\lim_{n\to\infty} \left| \frac{(n+1)^3}{3^{n+1}} \Big/ \frac{n^3}{3^n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)^3}{3^{n+1}} \cdot \frac{3^n}{n^3} \right| = \frac{1}{3} \lim_{n\to\infty} \left| \left(\frac{n+1}{n} \right)^3 \right| = \frac{1}{3} < 1.$$

(b) $\sum \frac{3^n}{n!}$ **Solution:** Converges by the ratio test, since

$$\lim_{n\to\infty} \left| \frac{3^{n+1}}{(n+1)!} \Big/ \frac{3^n}{n!} \right| = \lim_{n\to\infty} \left| \frac{3^{n+1}}{(n+1)!} \cdot \frac{n!}{3^n} \right| = 3 \lim_{n\to\infty} \left| \frac{1}{n+1} \right| = 0 < 1.$$

(d) $\sum \frac{n!}{(2^n)^3}$ **Solution:** Diverges by the ratio test, since

$$\lim_{n\to\infty} \left| \frac{(n+1)!}{(2^{n+1})^3} \Big/ \frac{n!}{(2^n)^3} \right| = \lim_{n\to\infty} \left| \frac{(n+1)!}{2^{3n+3}} \cdot \frac{2^{3n}}{n!} \right| = \lim_{n\to\infty} \left| \frac{n+1}{2^3} \right| = \infty.$$

(h) $\sum \frac{1}{n\sqrt{n+1}}$ **Solution:** Converges by the comparison test and the p-series test, since

$$\frac{1}{n\sqrt{n+1}} < \frac{1}{n^{3/2}}$$

for all $n \geq 1$, and $\sum 1/n^{3/2}$ converges.

(m) $\sum 2^n e^{-n}$ **Solution:** Converges by geometric series test, since $2^n e^{-n} = (2/e)^n$, and $|2/e| < 1$.

4. Determine the values of a for which each series converges.

Solution: We consider the integral

$$\int_2^k \frac{1}{x(\ln x)^p}\, dx.$$

To evaluate this integral, put $u = \ln x$. Then $du = dx/x$, so the integral becomes

$$\int_{\ln 2}^{\ln k} \frac{du}{u^p} = \begin{cases} \ln(k) - \ln(2) & \text{if } p = 1, \\ ((\ln k)^{1-p} - (\ln 2)^{1-p})/(1-p) & \text{if } p \neq 1. \end{cases}$$

From this, we see that

$$\lim_{k\to\infty} \int_2^k \frac{1}{x(\ln x)^p}\, dx$$

exists iff $p > 1$. Thus, by the integral test, the series in question converges iff $p > 1$.

5. Determine whether each series converges conditionally, converges absolutely, or diverges. Justify your answer.

(a) $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$ **Solution:** Converges by the alternating series test. But the series does not converge absolutely, since $1/\ln n > 1/n$ for all $n \geq 2$, and the harmonic series diverges. So $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$ converges conditionally.

(d) $\sum \frac{(-5)^n}{2^n}$ **Solution:** Diverges (and therefore, by Theorem 8.2.5, does not converge absolutely) by the geometric series test, since $(-5)^n/2^n = (-5/2)^n$, and $|-5/2| > 1$.

(i) $\sum_{n=2}^{\infty} \frac{(-1)^n \ln n}{n}$ **Solution:** Converges by the alternating series test. But the series does not converge absolutely, since $\ln n/n > 1/n$ for all $n \geq 2$, and the harmonic series diverges. So $\sum_{n=2}^{\infty} \frac{(-1)^n \ln n}{n}$ converges conditionally.

6. Find an example to show that the convergence of $\sum a_n$ and the convergence of $\sum b_n$ do not necessarily imply the convergence of $\sum(a_n b_n)$. (Compare with Exercise 8.1.11.) **Solution:** If $a_n = b_n = (-1)^n/\sqrt{n}$, then $\sum a_n$ and $\sum b_n$ converge by the alternating series test, but $\sum a_n b_n = \sum 1/n$, which diverges by the p-series test.