Assignment:

Section 7.2, pages 290–292: Exercises 1, 7, 10, 11.

Section 7.3, pages 298-300: Exercises 4, 5, 10, 15, 16.

Section 7.2:

1. Mark each statement True or False. Justify each answer.

(a) If f is monotone on $[a, b]$, then f is integrable on $[a, b]$. **Solution: True.** This is Theorem 7.2.1.

(b) If f is integrable on $[a, b]$, then f is continuous on $[a, b]$. **Solution: False.** Consider, for example, the function f of Example 7.2.3. This function is integrable on $[0, 1]$, as shown in that example, but is not continuous on $[0, 1]$, as shown in Example 5.2.9.

(c) If f and g are integrable on $[a, b]$ then $f + g$ is integrable on $[a, b]$, and $\int_a^b (f + g) = \int_a^b f + \int_a^b g$. **Solution: True.** This is Theorem 7.2.4(b).

7. Let $f : [0, 1] \to [0, 1]$ be the modified Dirichlet function of Example 7.2.3 and let $h : [0, 1] \to [0, 1]$ be the function of Example 7.1.8 (with domain restricted to $[0, 1]$). Find an integrable function $g : [0, 1] \to [0, 1]$ such that $h = g \circ f$, thereby showing that the composition of two integrable functions need not be integrable.

Solution: Let

$$g(x) = \begin{cases} 0 & \text{if } x = 0, \\ 1 & \text{if not.} \end{cases}.$$

Then g is monotone on $[0, 1]$, so by Theorem 7.2.1, it's integrable there. But then, if f is the modified Dirichlet function of Example 7.2.3, and $h = g \circ f$, we have

$$h(x) = g(f(x)) = \begin{cases} 0 & \text{if } f(x) = 0, \\ 1 & \text{if not} \end{cases} = \begin{cases} 0 & \text{if } x \text{ is irrational,} \\ 1 & \text{if } x \text{ is rational,} \end{cases}$$

since our function f equals 0 on the irrationals in $[0, 1]$ and is not equal to zero if $x \in [0, 1]$ is rational. So $h = g \circ f$ is the function of Example 7.1.8 (except with domain $[0, 1]$ instead of $[0, 2]$. We saw in that example that h is not integrable. So we have a compostion of two integrable functions that is not integrable.

10. Find an example of a function $f : [0, 1] \to \mathbb{R}$ such that f is not integrable on $[0, 1]$, but $|f|$ is integrable on $[0, 1]$. **Solution:** Let $f : [0, 1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} -1 & \text{if } x \text{ is irrational,} \\ 1 & \text{if } x \text{ is rational.} \end{cases}$$

By a slight modification of the argument in Example 7.1.7, we see that f is not integrable on $[0,1]$. But $|f(x)| = 1$ for all x, so $|f|$, being a constant function, is integrable on $[0,1]$.

11. Let f be integrable on $[a,b]$ and suppose that $m \le f(x) \le M$ for all $x \in [a,b]$. Show that $m(b-a) \le \int_a^b f \le M(b-a)$.

Solution: We've seen in class that, under the stated conditions, for any partition P of $[a,b]$,

$$m(b-a) \le L(f,P) \quad \text{and} \quad U(f,P) \le M(b-a).$$

By definition of $U(f)$ and $L(f)$, we also have, for any partition of $[a,b]$,

$$L(f,P) \le L(f) \quad \text{and} \quad U(f) \le U(f,P).$$

Moreover, if f is integrable on $[a,b]$, we have

$$L(f) = \int_a^b f = U(f).$$

Putting all of these inequalities together, we find that

$$m(b-a) \le L(f,P) \le L(f) = \int_a^b f = U(f) \le U(f,P) \le M(b-a),$$

which gives the desired result.

Section 7.3:

4. Let f be continuous on $[a,b]$. For each $x \in [a,b]$, let $F(x) = \int_x^b f$. Show that F is differentiable and that $F'(x) = -f(x)$. **Solution:** Write $\int_a^b f = \int_a^x f + \int_x^b f$. Differentiate both sides. The left hand side is constant with respect to x, so its derivative is zero. The derivative of $\int_a^x f$ with respect to x is $f(x)$, by Theorem 7.3.1. So we get $0 = f(x) + \frac{d}{dx}\int_x^b f$, or, solving, $\frac{d}{dx}\int_x^b f = -f(x)$.

5. Use Theorem 7.3.1 and the previous exercises to find a formula for the derivative of each function.

(a) $\int_0^x \sqrt{1+t^2}\,dt$. **Solution:** By Theorem 7.3.1, $\frac{d}{dx}\int_0^x \sqrt{1+t^2}\,dt = \sqrt{1+x^2}$.

(b) $\int_{-x}^x \sqrt{1+t^2}\,dt$. **Solution:** Write $\int_{-x}^x \sqrt{1+t^2}\,dt = \int_{-x}^0 \sqrt{1+t^2}\,dt + \int_0^x \sqrt{1+t^2}\,dt$. Using Theorem 7.3.1 combined with Exercise 4 above and Corollary 7.3.3, then, we find that

$$\begin{aligned}\frac{d}{dx}\int_{-x}^x \sqrt{1+t^2}\,dt &= \frac{d}{dx}\int_{-x}^0 \sqrt{1+t^2}\,dt + \frac{d}{dx}\int_0^x \sqrt{1+t^2}\,dt \\ &= -\sqrt{1+(-x)^2}\cdot\frac{d}{dx}(-x) + \sqrt{1+x^2} = \sqrt{1+x^2} + \sqrt{1+x^2} \\ &= 2\sqrt{1+x^2}.\end{aligned}$$

(c) $F(x) = \int_0^{\sin x} \cos t^2\, dt$. **Solution:** By Corollary 7.3.3,

$$\frac{d}{dx}\int_0^{\sin x} \cos t^2\, dt = \cos(\sin^2 x)\frac{d}{dx}\sin x = \cos x \cos(\sin^2 x).$$

(d) $\int_{x^2}^{x^3} \sqrt{1+t^2}\, dt$. **Solution:** Similarly to part (b), we have

$$\begin{aligned}\frac{d}{dx}\int_{x^2}^{x^3} \sqrt{1+t^2}\, dt &= \frac{d}{dx}\int_{x^2}^{0} \sqrt{1+t^2}\, dt + \frac{d}{dx}\int_0^{x^2} \sqrt{1+t^2}\, dt \\ &= -\sqrt{1+\left(x^2\right)^2}\cdot\frac{d}{dx}(x^2) + \sqrt{1+\left(x^3\right)^2}\cdot\frac{d}{dx}(x^3) \\ &= -2x\sqrt{1+x^4} + 3x^2\sqrt{1+x^6}.\end{aligned}$$

10. Use Theorem 7.3.1 to evaluate $\lim_{x\to 0}(1/x)\int_0^x \sqrt{9+t^2}\, dt$. **Solution:** By Theorem 7.3.1 and by l'Hôpital's rule,

$$\begin{aligned}\lim_{x\to 0}(1/x)\int_0^x \sqrt{9+t^2}\, dt &= \lim_{x\to 0}\frac{\int_0^x \sqrt{9+t^2}\, dt}{x} = \lim_{x\to 0}\frac{d\big(\int_0^x \sqrt{9+t^2}\, dt\big)/dx}{dx/dx} \\ &= \lim_{x\to 0}\frac{\sqrt{9}}{1} = 3.\end{aligned}$$

15. Use Exercise 14 to evaluate $\int_0^2 3x^2\sqrt{x^3+1}\, dx$. Identify the functions that you have used and explicitly write down $\int_0^2 3x^2\sqrt{x^3+1}\, dx$. **Solution:** Put $u = x^3+1$, so that $du = 3x^2\, dx$. Note that, when $x = 0$, $u = 0^3+1 = 1$, and when $x = 2$, $u = 2^3+1 = 9$. So

$$\int_0^2 3x^2\sqrt{x^3+1}\, dx = \int_1^9 \sqrt{u}\, du = \frac{u^{3/2}}{3/2}\bigg|_1^9 = \frac{2}{3}\big(9^{3/2} - 1^{3/2}\big) = \frac{2}{3}(27-1) = \frac{52}{3}.$$

16. Repeat Exercise 15 for $\int_0^{\pi/2}(\cos x)(1+\sin x)^3\, dx$. **Solution:** Put $u = 1+\sin x$, so that $du = \cos x\, dx$. Note that, when $x = 0$, $u = 1+\sin 0 = 1$, and when $x = \pi/2$, $u = 1+\sin(\pi/2) = 2$. So

$$\int_0^{\pi/2}(\cos x)(1+\sin x)^3\, dx = \int_1^2 u^3\, du = \frac{u^4}{4}\bigg|_1^2 = \frac{1}{4}\big(2^4-1^4\big) = \frac{1}{4}(16-1) = \frac{15}{4}.$$