Assignment:

Section 6.3, pages 263–266: Exercises 3abdegh, 4adefh, 5, 14.

Section 7.1, pages 281–283: Exercises 1, 3, 5, 8, 9.

Section 6.3:

3. Evaluate the following limits. (You may use the familiar derivative formulas for the elementary functions.)

(a) $\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{\cos x}{1} = \frac{\cos 0}{1} = 1.$

(b) $\lim_{x\to 0+} \frac{\sin x}{\sqrt{x}} = \lim_{x\to 0+} \frac{\cos x}{1/(2\sqrt{x})} = \lim_{x\to 0+} 2\sqrt{x}\cos x = 2\cdot\sqrt{0}\cdot\cos 0 = 0.$

(d) $\lim_{x\to 1} \frac{\cos(\pi x/2)}{x-1} = \lim_{x\to 0} \frac{-(\pi/2)\sin(\pi x/2)}{1} = \frac{-\pi\sin(\pi\cdot 1/2)}{2} = -\frac{\pi}{2}.$

(e) $\lim_{x\to 1} \frac{\ln x}{x-1} = \lim_{x\to 1} \frac{1/x}{1} = \frac{1/1}{1} = 1.$

(g)

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = \lim_{x\to\infty}\left(e^{\ln(1+1/x)}\right)^x = \lim_{x\to\infty} e^{x\ln(1+1/x)} = e^{\lim_{x\to\infty} x\ln(1+1/x)}. \qquad (*)$$

Now $\lim_{x\to\infty} x\ln(1+1/x)$ is of the form $\infty\cdot 0$. We convert it to the form $0/0$, and then apply l'Hôpital's:

$$\begin{aligned}\lim_{x\to\infty} x\ln\left(1+\frac{1}{x}\right) &= \lim_{x\to\infty}\frac{\ln(1+1/x)}{1/x} = \lim_{x\to\infty}\frac{1/(1+1/x)\cdot(-1/x^2)}{-1/x^2}\\ &= \lim_{x\to\infty}\frac{1}{1+1/x} = \frac{1}{1+0} = 1.\end{aligned}$$

So by $(*)$,

$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e^1 = e.$

(h) $\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{\sin x}\right) = \lim_{x\to 0}\frac{x-\sin x}{x\sin x} = \lim_{x\to 0}\frac{1-\cos x}{x\cos x+\sin x} = \lim_{x\to 0}\frac{\sin x}{\cos x - x\sin x + \cos x} = \frac{0}{1-0+1} = 0.$

4. Evaluate the following limits. (You may use the familiar derivative formulas for the elementary functions.)

(a)

$$\begin{aligned}\lim_{x\to 0}\frac{\tan x - x}{x^3} &= \lim_{x\to 0}\frac{\sec^2 x - 1}{3x^2}\\ &= \lim_{x\to 0}\frac{2\sec^2 x\tan x}{6x} = \lim_{x\to 0}\frac{\sin x}{3x\cos^3 x} = \lim_{x\to 0}\frac{\cos x}{9x\cos^2 x(-\sin x) + 3\cos^3 x}\\ &= \frac{1}{9\cdot 0\cdot \cos^2 0(-\sin 0) + 3\cos^3 0} = \frac{1}{3}.\end{aligned}$$

(d)

$$\lim_{x\to 0+}\frac{\ln\sin x}{\ln x} = \lim_{x\to 0+}\frac{\cos x/\sin x}{1/x} = \lim_{x\to 0+}\frac{x\cos x}{\sin x} = \lim_{x\to 0+}\frac{-x\sin x + \cos x}{\cos x} = \frac{-0\cdot 0 + 1}{1} = 1.$$

(e) $\displaystyle\lim_{x\to\infty}\frac{(\ln x)^2}{x} = \lim_{x\to\infty}\frac{2\ln x\cdot(1/x)}{1} = \lim_{x\to\infty}\frac{2\ln x}{x} = \lim_{x\to\infty}\frac{2/x}{1} = \frac{0}{1} = 0.$

(f) $\displaystyle\lim_{x\to 1}\frac{\ln x}{x^2+x-2} = \lim_{x\to\infty}\frac{1/x}{2x+1} = \frac{1/1}{2\cdot 1+1} = \frac{1}{3}.$

(h)

$$\begin{aligned}\lim_{x\to 0+}\frac{\ln x}{\csc x} &= \lim_{x\to 0+}\frac{1/x}{-\cot x\csc x} = \lim_{x\to 0+}\frac{-\tan x\sin x}{x} = \lim_{x\to 0+}\frac{-\sin^2 x}{x\cos x}\\ &= \lim_{x\to 0+}\frac{-2\sin x\cos x}{-x\sin x + \cos x} = \frac{-2\cdot 0\cdot 1}{-0\cdot 0 + 1} = 0.\end{aligned}$$

5. Indicate what is wrong with the following result:

$$\lim_{x\to 1}\frac{2x^2 - x - 1}{3x^2 - 5x + 2} = \lim_{x\to 1}\frac{4x-1}{6x-5} = \lim_{x\to 1}\frac{4}{6} = \frac{2}{3}.$$

Solution: Well first of all, the answer is wrong. But WHY is it wrong? What's wrong with the argument? The problem is that the second limit, $\lim_{x\to 1}\frac{4x-1}{6x-5}$, is not of the form $0/0$, so you can't apply l'Hôpital's. So at that point, you should just plug in $x = 1$, to get $(4\cdot 1 - 1)/(6\cdot 1 - 5) = 3/1 = 3$.

14. Suppose that h is continuous on $[a, b]$ and differentiable on (a, b), and that $c \in (a, b)$. Suppose also that $\lim_{x\to c} h'(x)$ exists. Prove that h' is continuous at c.

Solution: We're told that h is differentiable on (a, b) and that $c \in (a, b)$, so we know that $h'(c)$ exists. By definition of the derivative, we have

$$h'(c) = \lim_{x\to c}\frac{h(x) - h(c)}{x - c}. \qquad (**)$$

If we write $f(x) = h(x) - h(c)$ and $g(x) = x - c$, then $(**)$ reads

$$h'(c) = \lim_{x \to c} \frac{f(x)}{g(x)}. \qquad (**').$$

Moreover, note that f and g satisfy the hypotheses of l'Hôpital's rule (Theorem 6.3.2): both f and g are continuous on $[a, b]$ and differentiable on (a, b), $f(c) = h(c) - h(c) = 0$, $g(c) = c - c = 0$, and $g'(x) = 1$ is not equal to zero anywhere. So we can apply l'Hôpital's rule to $(**')$: we get

$$h'(c) = \lim_{x \to c} \frac{f'(x)}{g'(x)} = \lim_{x \to c} \frac{h'(x) - 0}{1} = \lim_{x \to c} h'(x). \qquad (**'').$$

But $(**'')$ is EXACTLY what it means for h' to be continuous at c, and we're done.

Section 7.1:

1. Let f be a bounded function defined on the interval $[a, b]$. Mark each statement True or False. Justify each answer.

(a) The upper and lower sums for f always form a bounded set. **Solution: True.** If P is any partition of $[a, b]$, then

$$m(b - a) \leq L(f, P) \leq U(f, P) \leq M(b - a),$$

where $m = \inf\{f(x) : x \in [a, b]\}$ and $M = \sup\{f(x) : x \in [a, b]\}$. So the set of all lower sums for f on $[a, b]$ is bounded below by $m(b - a)$ and above by $M(b - a)$, as is the set of upper sums for f on $[a, b]$.

(b) If P and Q are partitions of $[a, b]$, then $P \cup Q$ is a refinement of both P and Q. **Solution: True.** It says so right in Practice 7.1.5. Alternatively, simply note that $P \subseteq P \cup Q$ and $Q \subseteq P \cup Q$ always, and this is exactly what it means for $P \cup Q$ to be a refinement of both P and Q.

(c) f is Riemann integrable iff its upper and lower sums are equal. **Solution: False.** f is Riemann integrable iff its upper and lower *integrals* are equal. For a counterexample to the given statement, see, for example, Exercise 3 directly below.

3. Let $f(x) = x^2$ on $[0.5, 3]$.

(a) Find $L(f, P)$ and $U(f, P)$ when $P = \{0.5, 1, 2, 3\}$. **Solution:** Since f is increasing on $[0.5, 3]$, we know its minimum value on any interval occurs at the left endpoint of that interval, and its maximum value on any interval occurs at the right endpoint of that interval. So we have

$$\begin{aligned} L(f, P) &= f(0.5)(1 - 0.5) + f(1)(2 - 1) + f(2)(3 - 2) \\ &= 0.5^2 \cdot 0.5 + 1^2 \cdot 1 + 2^2 \cdot 1 = 0.125 + 1 + 4 = 5.125, \\ U(f, P) &= f(1)(1 - 0.5) + f(2)(2 - 1) + f(3)(3 - 1) \\ &= 1^2 \cdot 0.5 + 2^2 \cdot 1 + 3^2 \cdot 1 = 0.5 + 4 + 9 = 13.5. \end{aligned}$$

(b) Similarly, if $P = \{0.5, 1, 1.5, 2, 2.5, 3\}$, then all intervals defined by the partition have length equal to 0.5, so

$$\begin{aligned} L(f, P) &= 0.5 \cdot (f(0.5) + f(1) + f(1.5) + f(2) + f(2.5)) \\ &= 0.5(0.5^2 + 1^2 + 1.5^2 + 2^2 + 2.5^2) = 0.5 \cdot 13.75 = 6.875, \\ U(f, P) &= 0.5 \cdot (f(1) + f(1.5) + f(2) + f(2.5) + f(3)) \\ &= 0.5(1^2 + 1.5^2 + 2^2 + 2.5^2 + 3^2) = 0.5 \cdot 22.5 = 11.25. \end{aligned}$$

(c) $\displaystyle\int_{0.5}^{3} x^2\,dx = \frac{x^3}{3}\bigg|_{0.5}^{3} = \frac{3^3}{3} - \frac{0.5^3}{3} = 9 - 0.041\overline{6} = 8.958\overline{3}.$

5. Suppose that $f(x) = x$ for all $x \in [0, b]$. Show that f is integrable and that $\int_0^b f(x)\,dx = b^2/2$.

Solution: Let P_n be the partition of $[0, b]$ given by

$$P_n = \{0, b/n, 2b/n, \ldots nb/n = b\}.$$

We'll show that $L(f, P_n) = (n-1)b^2/(2n)$ and $U(f, P_n) = (n+1)b^2/(2n)$. Since $L(f, P_n) \le L(f) \le U(f) \le U(f, P_n)$ always, we'll have

$$\frac{(n-1)b^2}{2n} \le L(f) \le U(f) \le \frac{(n+1)b^2}{2n}. \qquad (***)$$

Noting that

$$\lim_{n\to\infty} (n-1)b^2/(2n) = \lim_{n\to\infty} (n+1)b^2/(2n) = \frac{b^2}{2},$$

it will follow from $(***)$ and from the squeeze law that $L(f) = U(f) = \int_0^b x\,dx = b^2/2$, which not only tells us that f is integrable, but evaluates the integral.

Note that the ith subinterval in our partition is the interval $[(i-1)b/n, ib/n]$. Since f is increasing on $[0, b]$, its minimum m_i on any such interval equals its value at the left endpoint $(i-1)b/n$ of that interval, and its maximum M_i on such an interval equals its value at the right endpoint ib/n of that interval. Note also that each subinterval has length b/n. So

$$\begin{aligned} L(f, P_n) &= \sum_{i=1}^{n} m_i \Delta x_i = \sum_{i=1}^{n} f\left(\frac{(i-1)b}{n}\right) \cdot \frac{b}{n} = \sum_{i=1}^{n} \frac{(i-1)b}{n} \cdot \frac{b}{n} \\ &= \frac{b^2}{n^2} \sum_{i=1}^{n} (i-1) = \frac{b^2}{n^2}(0 + 1 + \cdots + (n-1)) = \frac{(n-1)nb^2}{2n^2} = \frac{(n-1)b^2}{2n}, \end{aligned}$$

the next-to-last step by the formula from Example 3.1.3. Similarly,

$$\begin{aligned} U(f, P_n) &= \sum_{i=1}^{n} M_i \Delta x_i = \sum_{i=1}^{n} f\left(\frac{ib}{n}\right) \cdot \frac{b}{n} = \sum_{i=1}^{n} \frac{ib}{n} \cdot \frac{b}{n} \\ &= \frac{b^2}{n^2} \sum_{i=1}^{n} i = \frac{b^2}{n^2}(1 + 2 + \cdots + n) = \frac{n(n+1)b^2}{2n^2} = \frac{(n+1)b^2}{2n}. \end{aligned}$$

This is what we needed to show, and we're done.

8. Give an example of a function $f : [0, 1] \to \mathbb{R}$ that is not integrable on $[0, 1]$ but f^2 is integrable on $[0, 1]$. **Solution:** Let $f : [0, 1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} -1 & \text{if } x \text{ is irrational,} \\ 1 & \text{if } x \text{ is rational.} \end{cases}$$

By a very slight modification of the argument used in Example 7.1.8, we see that $L(f) = -1$ and $U(f) = 1$, so f is not integrable on $[0, 1]$. On the other hand, $f^2(x) = 1$ for all x, so clearly $L(f^2, P) = U(f^2, P) = 1$ for any partition P of $[0, 1]$, so $L(f^2) = U(f^2) = 1$, so f^2 is integrable on $[0, 1]$.

9. Prove or give a counterexample: If f and g are integrable on $[a, b]$ and h is a function such that $f(x) \leq h(x) \leq g(x)$ for all $x \in [a, b]$, then h is integrable on $[a, b]$. **Solution:** This is false. Let $f(x) = -1$ for all x, let $g(x) = 1$ for all x, and let $h(x)$ be the function of Exercise 8 directly above. Then f and g are integrable on $[0, 1]$ and h is a function such that $f(x) \leq h(x) \leq g(x)$ for all $x \in [0, 1]$, but h is not integrable on $[0, 1]$.