**Assignment:**

**Section 6.1, pages 245–248:** Exercises 1, 2, 4bd, 6abc, 12.

**Section 6.2, pages 255–258:** Exercises 1, 3, 5bd, 6, 10.

# Section 6.1:

**1.** Let $c$ be a point in the interval $I$ and suppose $f : I \to \mathbb{R}$. Mark each statement as True or False. Justify each answer.

(a) The derivative of $f$ at $c$ is defined by

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

wherever the limit exists. **Solution: False.** Strictly speaking, it should say "wherever the limit exists and is finite." As a counterexample, consider $f(x) = \sqrt[3]{x}$ and $c = 0$. We compute that

$$\lim_{x \to 0} \frac{f(x) - f(c)}{x - c} = \lim_{x \to 0} \frac{\sqrt[3]{x} - \sqrt[3]{0}}{x - 0} = \lim_{x \to 0} x^{-2/3} = +\infty.$$

So the limit *does* exist, at least according to the way things are worded in our book. See, for example, page 175, where it says:

> When $\lim s_n = +\infty$ (or $-\infty$), we shall say that the limit exists....

So, according to Definition 6.1.1, $f'(0)$ does not exist, in this case.

**Remark.** This problem amounts to a rather subtle question of language/terminology. Many would say that, if a limit is infinite, then that limit does *not* exist. Because of this, I'll accept either "true" or "false" for this exercise, though again, strictly speaking, in our text "the limit exists" does not necessarily mean the same as "the limit exists and is finite," so technically the answer here is **False.**

(b) If $f$ is continuous at $c$, then $f$ is differentiable at $c$. **Solution: False.** E.g. consider $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = |x|$. Then $f$ is continuous at 0 but not differentiable there.

(c) If $f$ is differentiable at $c$, then $f$ is continuous at $c$. **Solution: True.** This is Theorem 6.1.6.

**2.** Let $c$ be a point in the interval $I$ and suppose $f : I \to \mathbb{R}$. Mark each statement True or False. Justify each answer.

(a) If $f$ is differentiable at $c$, then for any $k \in \mathbb{R}$, $kf$ is differentiable at $c$. **Solution: True.** This is Theorem 6.1.7(a).

(b) Suppose $g : I \to \mathbb{R}$. If $f$ and $g$ are differentiable at $c$, then $f + g$ is differentiable at $c$. **Solution: True.** This is Theorem 6.1.7(b).

(c) Suppose $g : I \to \mathbb{R}$. If $f$ and $g$ are differentiable at $c$, then $g \circ f$ is differentiable at $c$. **False.** We need $g$ to be differentiable at $f(c)$. For example, let $I = \mathbb{R}$, and let $f(x) = x - 2$, $g(x) = |x|$. Then $f$ and $g$ are both differentiable at $c = 2$, but $g \circ f(x) = |x - 2|$ is not.

**4.** Use Definition 6.1.1 to find the derivative of each function.

(b) $f(x) = x^3$ for $x \in \mathbb{R}$. **Solution:** For $c \in \mathbb{R}$, we have

$$\begin{aligned} f'(c) &= \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \\ &= \lim_{x \to c} \frac{x^3 - c^3}{x - 2} = \lim_{x \to c} \frac{(x - c)(x^2 + cx + c^2)}{x - c} = \lim_{x \to c}(x^2 + cx + c^2) \\ &= c^2 + c^2 + c^2 = 3c^2. \end{aligned}$$

(d) $f(x) = \sqrt{x}$ for $x > 0$. **Solution:** For $c \in (0, \infty)$, we have

$$\begin{aligned} f'(c) &= \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \\ &= \lim_{x \to c} \frac{\sqrt{x} - \sqrt{c}}{x - c} = \lim_{x \to c} \frac{\sqrt{x} - \sqrt{c}}{x - c} \cdot \frac{\sqrt{x} + \sqrt{c}}{\sqrt{x} + \sqrt{c}} \\ &= \lim_{x \to c} \frac{x - c}{(x - c)(\sqrt{x} + \sqrt{c})} = \lim_{x \to c} \frac{1}{\sqrt{x} + \sqrt{c}} \\ &= \frac{1}{\sqrt{c} + \sqrt{c}} = \frac{1}{2\sqrt{c}}. \end{aligned}$$

**6.** Let $f(x) = x^2 \sin(1/x)$ for $x \neq 0$ and $f(0) = 0$.

(a) Use the chain rule and the product rule to show that $f$ is differentiable at each $c \neq 0$ and find $f'(c)$. (You may assume that the derivative of $\sin x$ is $\cos x$ for all $x \in \mathbb{R}$.)

**Solution:** For $x \neq 0$,

$$\begin{aligned} f'(x) &= x^2 \frac{d}{dx} \sin\left(\frac{1}{x}\right) + \sin\left(\frac{1}{x}\right) \frac{d}{dx} x^2 \\ &= x^2 \cos\left(\frac{1}{x}\right) \cdot \frac{d}{dx}\left(\frac{1}{x}\right) + 2x \sin\left(\frac{1}{x}\right) \\ &= x^2 \cos\left(\frac{1}{x}\right) \cdot \left(\frac{-1}{x^2}\right) + 2x \sin\left(\frac{1}{x}\right) = -\cos\left(\frac{1}{x}\right) + 2x \sin\left(\frac{1}{x}\right). \end{aligned}$$

So of course $f'(c) = -\cos(1/c) + 2c \sin(1/c)$ for $c \neq 0$.

(b) Use Definition 6.1.1 to show that $f$ is differentiable at $c = 0$ and find $f'(0)$. **Solution:** We have

$$\begin{aligned} f'(c) &= \lim_{x\to c} \frac{f(x) - f(c)}{x - c} \\ &= \lim_{x\to c} \frac{x^2 \sin(1/x) - 0}{x - 0} = \lim_{x\to c} x \sin\left(\frac{1}{x}\right) = 0, \end{aligned}$$

by the squeeze law.

(c) Show that $f'$ is not continuous at $x = 0$. **Solution:** If $f'$ were continuous at $x = 0$, we would have $\lim_{x\to 0} f'(x) = f'(0)$, by Theorem 5.2.2(a$\Rightarrow$d). But $f'(0) = 0$ by part (b) above, while $\lim_{x\to 0} f'(x)$ does not exist, by part (a) above. (Although $\lim_{x\to 0} 2x \sin(1/x)$ does exist, and equals zero by the squeeze law, $\lim_{x\to 0} \cos(1/x)$ does not exist, by the same reasoning as we used for Example 5.1.11.) So $f'$ is not continuous at $x = 0$.

**12.** Prove: if a polynomial $p(x)$ is divisible by $(x - a)^2$, then $p'(x)$ is divisible by $(x - a)$. **Solution:** First of all, it follows from Example 6.1.8 that all polynomials are differentiable at all points in $\mathbb{R}$.

Suppose $p(x)$ is a polynomial divisible by $(x - a)^2$. This means $p(x) = (x - a)^2 q(x)$ for some polynomial $q(x)$. Applying the product rule, together with the chain rule (to differentiate $(x - a)^2$), then gives

$$\begin{aligned} p'(x) &= (x - a)^2 q'(x) + q(x) \frac{d}{dx}(x - a)^2 \\ &= (x - a)^2 q'(x) + 2(x - a)q(x) = (x - a)\big((x - a)q'(x) + 2q(x)\big). \end{aligned}$$

Since $q(x)$ is a polynomial, so is $(x - a)q'(x) + 2q(x)$. So $p'(x)$ is divisible by $(x - a)$. □

# Section 6.2:

**1.** Mark each statement as True or False. Justify each answer.

(a) A continuous function defined on a bounded interval assumes maximum and minimum values. **Solution: False.** This need not be true if the interval is not compact. For example, $f(x) = x$ on $(0, 1)$ does not attain a maximum or a minimum there.

(b) If $f$ is continuous on $[a, b]$, then there exists a point $c \in (a, b)$ such that $f'(c) = [f(b) - f(a)]/(b - a)$. **Solution: False.** This need not be true if $f$ is not differentiable on $(a, b)$. For example, $f(x) = |x|$ is continuous on $[-1, 1]$, but there is no point $c$ in $(-1, 1)$ where $f'(c) = [f(1) - f(-1)]/(1 - (-1)) = 0$.

(c) Suppose $f$ is differentiable on $(a, b)$. If $c \in (a, b)$ and $f'(c) = 0$, then $f(c)$ is either the maximum or the minimum of $f$ on $(a, b)$. **Solution: False.** Consider $f(x) = x^3$ on $(-1, 1)$. Certainly $f$ is differentiable on this interval. We have $f'(0) = 0$, but $f(0) = 0$ is neither the maximum nor the minimum value of $f$ on this interval.

**3.** Let $f(x) = x^2 - 4x + 5$ for $x \in [0, 3]$.

(a) Find where $f$ is strictly increasing and where it is strictly decreasing. **Solution:** We have $f'(x) = 2x - 4$. By Theorem 6.2.8, $f$ is strictly increasing if $2x - 4 > 0$, meaning $x > 2$, meaning $x \in (2, 3]$, since our domain is $[0, 3]$. Similarly, $f$ is strictly decreasing if $2x - 4 < 0$, meaning $x < 2$, meaning $x \in [0, 2)$.

(b) Find the maximum and minimum of $f$ on $[0, 3]$. **Solution:** Since $f$ is decreasing on $[0, 2)$ and increasing on $(2, 3]$, its minimum value, on $[0, 3]$, must occur at $x = 2$. This minimum value is $f(2) = 1$. Its maximum value must then be at one of the endpoints. We have $f(0) = 5$ and $f(3) = 2$. Since 5 is the larger of these numbers, we find that the maximum value of $f$, on $[0, 3]$, is $f(0) = 5$.

**5.** Use the mean value theorem to establish the following inequalities. (You may assume any relevant derivative formulas from calculus.)

(b) $\dfrac{x-1}{x} < \ln x < x - 1$ for $x > 1$. **Solution:** Let $f(t) = \ln t$, on the interval $[1, x]$. Since $f$ is continuous on this interval and differentiable on $(1, x)$, the mean value theorem tells us that, for some number $c \in (1, x)$,

$$\frac{f(x) - f(1)}{x - 1} = f'(c) = \frac{1}{c}.$$

That is,

$$\frac{\ln x}{x - 1} = \frac{1}{c}. \qquad (*)$$

But since $c \in (1, x)$, we have $c < x$, so $1/c > 1/x$, so $(*)$ gives

$$\frac{\ln x}{x - 1} > \frac{1}{x},$$

or $\ln x > \dfrac{x-1}{x}$, as claimed. Moreover, since $c \in (1, x)$, we have $c > 1$, so $1/c < 1$, so $(*)$ gives

$$\frac{\ln x}{x - 1} < 1,$$

or $\ln x < x - 1$, also as claimed.

(d) $\sqrt{1 + x} < 5 + \dfrac{x - 24}{10}$ for $x > 24$. **Solution:** Let $f(t) = \sqrt{1 + t}$, on the interval $[24, x]$. Since $f$ is continuous on this interval and differentiable on $(24, x)$, the mean value theorem tells us that, for some number $c \in (24, x)$,

$$\frac{f(x) - f(24)}{x - 24} = f'(c) = \frac{1}{2\sqrt{1 + c}}.$$

That is,

$$\frac{\sqrt{1+x}-\sqrt{25}}{x-24} = \frac{1}{2\sqrt{1+c}}. \tag{$**$}$$

But since $c \in (24, x)$, we have $c > 24$, so $(**)$ gives

$$\frac{\sqrt{1+x}-\sqrt{25}}{x-24} < \frac{1}{2\sqrt{1+24}},$$

or

$$\frac{\sqrt{1+x}-5}{x-24} < \frac{1}{10},$$

or $\sqrt{1+x} < 5 + \dfrac{x-24}{10}$, as claimed.

**6.** Rolle's theorem requires three conditions be satisfied:

(i) $f$ is continuous on $[a, b]$,

(ii) $f$ is differentiable on $[a, b]$, and

(iii) $f(a) = f(b)$.

Find three functions that satisfy two of these conditions, but for which the conclusion of Rolle's theorem does not follow. That is, there is no point $c \in (a, b)$ such that $f'(c) = 0$.

**Solution:** The function $f(x) = x$ satisfies the first two conditions on $[1, 5]$, but not the third. Note that there is no $c \in (1, 5)$ with $f'(c) = 0$. Next: the function $f(x) = |x|$ satisfies the first and third conditions on $[-1, 1]$, but not the second. Note that there is no $c \in (-1, 1)$ with $f'(c) = 0$. Finally: if $f(-1) = 3$ and $f(x) = x$ on $(-1, 3]$, then $f$ satisfies the second and third conditions on $[-1, 3]$, but not the first. Note that there is no point $c \in (-1, 3)$ with $f'(c) = 0$.

**10.** Let f be differentiable on $(0, 1)$ and continuous on $[0, 1]$. Suppose that $f(0) = 0$ and that $f'$ is increasing on $(0, 1)$. (See Exercise 8.) Let $g(x) = f(x)/x$ for $x \in (0, 1)$. Prove that $g$ is increasing on $(0, 1)$.

**Solution:** It suffices to show that $g' > 0$ on $(0, 1)$. To do so, we use the quotient rule to show that, on $(0, 1)$,

$$g'(x) = \frac{d}{dx}\left(\frac{f(x)}{x}\right) = \frac{xf'(x) - f(x)}{x^2}.$$

Note that the denominator is always positive on $(0, 1)$, so to show that $g'(x)$ is always positive on $(0, 1)$, it suffices to show that the numerator $xf'(x) - f(x)$ is always positive on $(0, 1)$. That is, we need only show that, for $x \in (0, 1)$, we have $xf'(x) - f(x) > 0$, or

$$f'(x) > \frac{f(x)}{x}. \tag{$*'$}$$

(We have divided through by $x$. This doesn't change the direction of the inequality, because we're assuming $x > 0$.)

To show $(*')$, for $x \in (0,1)$, we apply the mean value theorem to the function $f$ on the interval $(0,x)$. That theorem tells us that there is a number $c$ in this interval with

$$f'(c) = \frac{f(x) - f(0)}{x - 0} = \frac{f(x)}{x}. \tag{$\dagger$}$$

Now since $c \in (0,x)$, we have $x > c$. But we're assuming that $f'$ is increasing on $(0,1)$, so $f'(x) > f'(c)$. So by $(\dagger)$,

$$f'(x) > \frac{f(x)}{x},$$

which is what we wanted to show, and we're done. $\square$