Assignment:

Section 5.1, pages 203–205: Exercises 1, 5, 6c.

Section 5.2, pages 212–214: Exercises 1, 2, 6, 13.

Section 5.3, pages 220–221: Exercises 1, 3abdef (hint: they're all false!), 11.

Section 5.1:

1. Let $f : D \colon \mathbb{R}$ and let c be an accumulation point of D. Mark each statement as True or False. Justify each answer.

(a) $\lim_{x\to c} f(x) = L$ iff for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - L| < \varepsilon$ whenever $x \in D$ and $|x - c| < \delta$. **Solution: False.** Instead of "$|x - c| < \delta$," it should say "$0 < |x - c| < \delta$." For example, let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x$ if $x \neq 1$, but $f(1) = 2$. Let $c = 1$ and $L = 1$. It's true that $\lim_{x\to c} f(x) = L$, but it's not true that, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $x \in \mathbb{R}$ and $|x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon$. For example, let $\varepsilon = 1/2$. Then, given *any* $\delta > 0$, let $x = 1$. Then certainly $|x - c| = |1 - 1| = 0 < \delta$, but

$$|f(x) - L| = |f(1) - 1| = |2 - 1| = 1 > \frac{1}{2} = \varepsilon.$$

(b) $\lim_{x\to c} f(x) = L$ iff for every deleted neighborhood U of c there exists a neighborhood V of L such that $f(U \cap D) \subseteq V$. **Solution: False.** This is not what Theorem 5.1.2 says. In that Theorem, instead of "for every deleted neighborhood U of c there exists a neighborhood V of L...," it says "for every neighborhood V of L there exists a deleted neighborhood U of c...."

Of course, the fact that Theorem 5.1.2 doesn't say something does not mean that something is false. (For example, Theorem 5.1.2 does not say "All squares have four sides," yet all squares *do* have four sides.) To *prove* that the statement is false, we need a counterexample. Here's one: Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = \tan(x)$. Let $c = 0$ and $L = 0$. Then certainly $\lim_{x\to c} f(x) = L$. But now take the deleted neighborhood $U = N^*(0, \frac{\pi}{2})$ of c. Then

$$f(U \cap D) = f\left(N^*\left(0, \frac{\pi}{2}\right)\right) = (-\infty, \infty) = \mathbb{R}$$

(the image of the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$ under the tangent function is the entire real line). So $f(U \cap D)$ is not contained in any neighborhood V of L, since a neighborhood is of the form $N(x, \varepsilon)$ for some *finite* number ε, so neighborhoods are bounded, but $f(U \cap D) = \mathbb{R}$ is not.

(c) $\lim_{x\to c} f(x) = L$ iff for every sequence (s_n) in D that converges to c with $s_n \neq c$ for all n, the sequence $(f(s_n))$ converges to L. **Solution: True.** This is exactly the statement of Theorem 5.1.8.

(d) If f does not have a limit at c, then there exists a sequence (s_n) in D with each $s_n \neq c$ such that (s_n) converges to c, but $(f(s_n))$ is divergent. **Solution: True.** This is exactly the statement of Theorem 5.1.10.

5. Find a $\delta > 0$ so that $|x+2| < \delta$ implies that $|x^2 - 3x - 10| < 1/3$. **Solution:** Let $\varepsilon = 1/3$. [Scratchwork: we want to express $|x^2 - 3x - 10|$ in terms of $|x+2|$. We have

$$|x^2-3x-10| = |(x+2)(x-5)| = |(x+2)(x+2-7)| = |x+2|\,|x+2-7| \leq |x+2|\,\big(|x+2|+7\big).$$

We want this to be $< 1/3$. Let's first require that $|x+2| < 1$. Then we have $|x+2|\,\big(|x+2|+7\big) < |x+2|(1+7) = 8|x+2|$. If we now also require that $|x+2| < (1/3)/8 = 1/24$, then we get $8|x+2| < 8 \cdot (1/24) = 1/3$, which is what we want. Of course $1/24$ is less than 1, so $|x+2| < 1/24$ will work. So this is what we write.] Let $\delta = 1/24$. Then

$$\begin{aligned}|x+2| < \delta \Rightarrow |x^2-3x-10| &= |(x+2)(x-5)| \\ &= |(x+2)(x+2-7)| = |x+2|\,|x+2-7| \leq |x+2|\,\big(|x+2|+7\big) \\ &< \frac{1}{24} \cdot (1+7) = 1/3.\end{aligned}$$

6c. Use Definition 5.1.1 to prove that

$$\lim_{x\to 2} x^3 = 8.$$

Solution: Let $\varepsilon > 0$. [Scratchwork: using long division of polynomials, you can show that $x^3 - 8 = (x-2)(x^2+2x+4)$. But then, again using long division of polynomials, you can show that $x^2 + 2x + 4 = (x-2)(x+4) + 12$. Note that $x + 4 = x - 2 + 6$. So

$$\begin{aligned}|x^3 - 8| &= |(x-2)|\,|x^2+2x+4| = |(x-2)|\,|(x-2)(x-2+6)+12| \\ &\leq |x-2|\,\big(|x-2|\,(|x-2|+6)+12\big).\end{aligned}$$

If we first assume that $|x-2| < 1$, then the right-hand side of the above is $< |x-2| \cdot (1\cdot(1+6)+12) = 19|x-2|$. So if we also assume that $|x-2| < \varepsilon/19$, we have what we want. So this is what we write.] Let $\delta = \min\{1, \varepsilon/19\}$. Then

$$\begin{aligned}|x-2| < \delta \Rightarrow |x^3-8| &= |(x-2)|\,|x^2+2x+4| = |(x-2)|\,|(x-2)(x-2+6)+12| \\ &\leq |x-2|\,\big(|x-2|\,(|x-2|+6)+12\big) < |x-2|(1\cdot(1+6)+12) \\ &= 19|x-2| < 19 \cdot \frac{\varepsilon}{19} = \varepsilon.\end{aligned}$$

So $\lim_{x\to 2} x^2 = 8$.

Section 5.2:

1. Let $f : D \to \mathbb{R}$ and let $c \in D$. Mark each statement as True or False. Justify each answer.

(a) f is continuous at c iff for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - f(c)| < \varepsilon$ whenever $|x - c| < \delta$ and $x \in D$. **Solution: True.** This is Definition 5.2.1.

(b) If $f(D)$ is a bounded set, then f is continuous on D. **Solution: False.** Let $f \colon \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = -1$ if $x \leq 0$ and $f(x) = 1$ if $x > 0$. Then $f(D)$ is bounded ($f(D) = f(\mathbb{R}) = \{-1, 1\}$), but f is not continuous at $c = 0$.

(c) If c is an isolated point of D, then f is continuous at c. **Solution: True.** We stated this in class, and gave the example of $f \colon \mathbb{N} \to \mathbb{R}$ given by $f(n) = n^2$. We showed that f is continuous at $c = 3$. As we noted, that example can be generalized to show that any function is continuous at any isolated point in its domain. See also p. 206 of the text.

(d) If f is continuous at c and (x_n) is a sequence in D, then $x_n \to c$ whenever $f(x_n) \to f(c)$. **Solution: False.** For example, consider $f \colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \sin x$. Certainly f is continuous on $\mathbb{R}$. Let $x_n = \pi + \frac{1}{n}$ and let $c = 0$. Then $(f(x_n)) = (\sin(\pi + \frac{1}{n}))$ converges to $f(c) = f(0) = 0$, even though $x_n \to \pi \neq 0$.

(e) If f is continuous at c, then for every neighborhood V of $f(c)$ there exists a neighborhood U of c such that $f(U \cap D) = V$. **Solution: False.** The conclusion of Theorem 5.2.2(a$\Rightarrow$c) says "$f(U \cap D) \subseteq V$," not "$f(U \cap D) = V$." Here's a counterexample: let $f(x) = x^2$, with domain $\mathbb{R}$. This function is continuous at $c = 0$, and $f(0) = 0$. Now take any neighborhood $N(0, \varepsilon) = (-\varepsilon, \varepsilon)$ of $f(0)$. No neighborhood U of 0 can satisfy $f(U \cap D) = V$, because if U is a neigborhood of 0, then $U = N(0, \delta)$ for some real number δ, so $f(U \cap D) = f(U) = f((-\delta, \delta)) = [0, \delta^2)$, which cannot equal V, because V contains negative numbers.

2. Let $f : D \to \mathbb{R}$ and let $c \in D$. Mark each statement as True or False. Justify each answer.

(a) If f is continuous at c and c is an accumulation point of D, then $\lim_{x \to c} f(x) = f(c)$. **Solution: True.** This is Theorem 5.2.2(a$\Rightarrow$d).

(b) Every polynomial is continuous at each point in $\mathbb{R}$. **Solution: True.** This is Example 5.2.3.

(c) If x_n is a Cauchy sequence in D, then $(f(x_n))$ is convergent. **Solution: False.** For example, if $x_n = 1/n$, then (x_n) is convergent and therefore Cauchy. But if $f(x) = 1/x$, then $f(x_n) = 1/(1/n) = n$ does not define a convergent sequence.

(d) If $f \colon \mathbb{R} \to \mathbb{R}$ is continuous at each irrational number, then f is continuous on $\mathbb{R}$. **Solution: False.** See Example 5.2.9.

(e) If $f\colon \mathbb{R} \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$ are both continuous (on $\mathbb{R}$), then $f \circ g$ and $g \circ f$ are both continuous on $\mathbb{R}$. **Solution: True.** This follows from Theorem 5.2.12. We do have to check that $f(\mathbb{R})$ is contained in the domain of g and that $g(\mathbb{R})$ is contained in the domain of f, but this is clear, since both domains equal $\mathbb{R}$.

6. Prove or give a counterexample for each statement.

(a) If f is continuous on D and $k \in \mathbb{R}$, then kf is continuous on D. **Solution:** This is true. Proof: this is automatic at any isolated point of D, because all functions are continuous at such a point. So suppose c is an accumulation point of D. By Theorem 5.2.2(a⇒d), $\lim_{x\to c} f(x) = f(c)$. Then by Theorem 5.1.13, $\lim_{x\to c} kf(x) = kf(c)$. Then by Theorem 5.2.2(d⇒a), kf is continuous at c.

(b) If f and $f + g$ are continuous on D, then g is continuous on D. **Solution:** This is true, by Theorem 5.2.10 and the fact that $g = (f + g) + (-f)$. (Strictly speaking, we need to show that f is continuous whenever f is, but this follows from part (a) of this exercise with $k = -1$.)

(c) If f and fg are continuous on D, then g is continuous on D. **Solution:** This is false. E.g. define functions f and g on $\mathbb{R}$ by $f(x) = x^2$, and $g(x) = 1/x$ for $x \neq 0$, but $g(0) = 1$. Note that $fg(x) = x$ for $x \neq 0$, and $fg(0) = 0 \cdot 1 = 0$. So $fg(x) = x$ for all x, so fg is continuous on $\mathbb{R}$. So is f, but g is not.

(d) If f^2 is continuous on D, then f is continuous on D. **Solution:** This is false. Let f be as in Exercise 1(b), Section 5.2 (see above). Then f^2 is continuous on $\mathbb{R}$ ($f^2(x) = 1$ for all $x \in \mathbb{R}$), but f is not.

(e) If f is continuous on D and D is bounded, then $f(D)$ is bounded. **Solution:** This is false. E.g. let $D = (0, 1)$ and $f(x) = 1/x$.

(f) If f and g are not continuous on D, then $f + g$ is not continuous on D. **Solution:** This is false. E.g. let f be the function of Exercise 1(b), Section 5.2 (see above), and let $g = -f$. Then neither f nor g is continuous on $\mathbb{R}$, but $f+g$ is, since $(f+g)(x) = 0$ for all x.

(g) If f and g are not continuous on D, then fg is not continuous on D. **Solution:** This is false. E.g. let f and g both be the function of Exercise 1(b), Section 5.2 (see above). Then neither f nor g is continuous on $\mathbb{R}$, but fg is, since $(fg)(x) = 1$ for all x.

(h) If $f\colon D \to E$ and $g : E \to F$ are not continuous on D and E respectively, then $g \circ f : D \to F$ is not continuous on D. **Solution:** This is false. For example, Let

$$f(x) = \begin{cases} 1/x^2 & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

and let

$$g(x) = \begin{cases} 1/x & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Clearly, both functions are discontinuous at $x = 0$. but continuous elsewhere. But note that

$$g(f(x)) = \begin{cases} g(1/x^2) = x^2 & \text{if } x \neq 0, \\ g(f(0)) = g(0) = 0 & \text{if } x = 0. \end{cases}$$

So $g \circ f$ is continuous at $x = 0$ (and, clearly, everywhere else).

13. Let $f : D \to \mathbb{R}$ be continuous at $c \in D$ and suppose that $f(c) > 0$. Prove that there exists an $\alpha > 0$ and a neigborhood U of c such that $f(x) > \alpha$ for all $x \in U \cap D$.

Solution: Let $\epsilon = f(c)/2$. We're assuming $f(c) > 0$, so $\epsilon > 0$. By definition of continuity, there is a $\delta > 0$ such that $x \in D$ and $|x - c| < \delta$ implies $|f(x) - f(c)| < \epsilon = f(c)/2$. But note that $f(c) - f(x) \leq |f(x) - f(c)|$ always. So $x \in D$ and $|x - c| < \delta$ implies $f(c) - f(x) < f(c)/2$ or, solving for $f(x)$, $f(x) > f(c)/2$. But note that saying $x \in D$ and $|x - c| < \delta$ is the same as saying that x is in $U \cap D$, where U is the neighborhood $N(c, \delta)$ of c. So we've found a neighborhood U of c such that $x \in U \cap D \Rightarrow f(x) > \alpha = f(c)/2$.

Section 5.3:

1. Mark each statement as true or false. Justify each answer.

(a) Let D be a compact subset of $\mathbb{R}$ and suppose that $f\colon D \to \mathbb{R}$ is continuous. Then $f(D)$ is compact. **Solution: True.** This is exactly Theorem 5.3.2.

(b) Suppose that $f\colon D \to \mathbb{R}$ is continuous. Then there exists a point $x_1 \in D$ such that $f(x_1) \geq f(x)$ for all $x \in D$. **Solution: False.** The function $f(x) = x$ is continuous on $D = \mathbb{R}$, but there is no real number x_1 such that $f(x_1) \geq f(x)$ for all $x \in \mathbb{R}$.

(c) Let D be a bounded subset of $\mathbb{R}$ and suppose that $f\colon D \to \mathbb{R}$ is continuous. Then $f(D)$ is bounded. **Solution: False.** For example, $f(x) = 1/x$ is continuous on $(0, 1)$, and $(0, 1)$ is bounded, but $f((0, 1)) = (1, \infty)$ is not.

3. Let $f\colon D \to \mathbb{R}$ be continuous. For each of the following, prove or give a counterexample.

(a) If D is open, then $f(D)$ is open. **Solution:** This is false. For example, Let $f(x) = \sin x$ and $D = (-10, 10)$ (or $D = \mathbb{R}$). Then D is open, but $f(D) = [-1, 1]$ is not.

(b) If D is closed, then $f(D)$ is closed. **Solution:** This is false. For example, Let $f(x) = \arctan(x)$ and $D = \mathbb{R}$. Then D is closed, but $f(D) = (-\frac{\pi}{2}, \frac{\pi}{2})$ is not.

(d) If D is not closed, then $f(D)$ is not closed. **Solution:** This is false: see part (a) of this exercise, above.

(e) If D is not compact, then $f(D)$ is not compact. **Solution:** This is false: see part (a) of this exercise, above.

(f) If D is unbounded, then $f(D)$ is unbounded. **Solution:** This is false. For example, Let $f(x) = \sin x$ and $D = \mathbb{R}$. Then D is unbounded, but $f(D) = [-1, 1]$ is not.

11. (a) Let $p \in \mathbb{R}$ and define $f\colon \mathbb{R} \to \mathbb{R}$ by $f(x) = |x - p|$. Prove that f is continuous.

Solution: Let $c \in \mathbb{R}$ and let $\varepsilon > 0$. Let $\delta = \varepsilon$. Then, since $\big||x| - |y|\big| \leq |x - y|$ always (see Exercise 6a, Section 3.2), we have

$$\begin{aligned} |x - c| < \delta \Rightarrow |f(x) - f(c)| &= \Big||x - p| - |x - c|\Big| \\ &\leq \Big|(x - p) - (c - p)\Big| = |x - c| < \delta = \varepsilon. \end{aligned}$$

So f is continuous at c. □

(b) Let S be a compact subset of $\mathbb{R}$ and let $p \in \mathbb{R}$. Prove that S has a "closest point" to p. That is, prove that there exists a point $q \in S$ such that $|q - p| = \inf\{|x - p| : x \in S\}$.

Solution: Let S and p be as stated; let f be as in part (a) of this exercise. Since f is continuous on S and S is, by assumption, compact, we know by corollary 5.3.3 that there is a point $q \in S$ with $f(q) \leq f(x)$ for all $x \in S$. That is, by definition of f, $|q-p| \leq |x-p|$ for all $x \in S$.