Assignment:

Section 3.4, pages 140-143: Exercises 1acdgi, 2abefh, 7adeg, 9a, 10, 14.

Section 3.5, pages 148-151: 1, 2, 3c, 4, 8b.

Section 4.1, pages 169-170: 1, 2, 3bc, 6cd, 13 (HINT for #13: see S-POP p. 15, Proposition B(iv)-1_{EEEEE}.)

Section 3.4:

1. Let $S \subset \mathbb{R}$. Mark each statement as True or False. Justify each answer.

(a) $\operatorname{int} S \cap \operatorname{bd} S = \emptyset$. **Solution: True.** If $x \in \operatorname{int} S$, then some neighborhood $N(x, \varepsilon)$ of x is contained completely in S. But then x can't be in $\operatorname{bd} S$, because *every* neighborhood of a boundary point of S intersects $\mathbb{R}\backslash S$.

(c) $\operatorname{bd} S \subseteq S$. **Solution: False.** For example, if $S = (0, 1)$, then $0 \in \operatorname{bd} S$ but $0 \notin S$.

(d) S is open iff $S = \operatorname{int} S$. **Solution: True.** This is Theorem 3.4.7(a).

(g) Every neighborhood is an open set. **Solution: True.** By Definition 3.4.1, a neighborhood of x is a set of the form $(x - \varepsilon, x + \varepsilon)$, which is an open interval, and thus an open set.

(i) The union of any collection of closed sets is closed. **Solution: False.** For example, the union of the collection $\{[\frac{1}{n}, 1 - \frac{1}{n}] \colon n \in \mathbb{N}\}$ of closed intervals equals $(0, 1)$, which is not closed.

7. Let S and T be subsets of $\mathbb{R}$. Find a counterexample for each of the following.

(a) If P is the set of all isolated points of S, then P is a closed set. **Solution:** A counterexample is the set $S = \{1 + \frac{1}{n} \colon n \in \mathbb{N}\}$. Note that every point in S is an isolated point of S, so the set of isolated points of S **is** S. But S is not closed, because it does not contain the point 1, which is a boundary point of S.

(d) If S is open, then $\operatorname{int}(\operatorname{cl} S)) = S$. **Solution:** A counterexample is the set $S = (0, 1) \cup (1, 2)$. We have $\operatorname{cl} S = [0, 2]$, so $\operatorname{int}(\operatorname{cl} S)) = \operatorname{int}[0, 2] = (0, 2) \neq S$.

(e) $\operatorname{bd}(\operatorname{cl} S) = \operatorname{bd} S$. **Solution:** A counterexample is the set $S = [0, 1) \cup (1, 2]$. We have $\operatorname{cl} S = [0, 2]$, so $\operatorname{bd}(\operatorname{cl} S) = \{0, 2\}$, while $\operatorname{bd} S = \{0, 1, 2\}$.

(g) $\operatorname{bd}(S \cup T) = (\operatorname{bd} S) \cup (\operatorname{bd} T)$. **Solution:** A counterexample is given by the sets $S = [0, 1]$ and $T = [1, 2]$. We have $\operatorname{bd}(S \cup T) = \operatorname{bd}[0, 2] = \{0, 2\}$, while $(\operatorname{bd} S) \cup (\operatorname{bd} T) = \{0, 1\} \cup \{1, 2\} = \{0, 1, 2\}$.

9. Prove the following. (a) An accumulation point of a set S is either an interior point of S or a boundary point of S.

Solution: Let x be an accumulation point of a set S. If $x \in \operatorname{int} S$, then we're done. If not, then we must show that $x \in \operatorname{bd} S$. By definition of boundary point, this means: we must show that any neighborhood $N(x, \varepsilon)$ of x intersects both S and $\mathbb{R} \backslash S$.

So let $N(x, \varepsilon)$ be such a neighborhood. Since x is an accumulation point of S we know, by definition of accumulation point, that $N^*(x, \varepsilon)$ intersects S; since $N^*(x, \varepsilon) \subseteq N(x, \varepsilon)$, we conclude that $N(x, \varepsilon)$ intersects S as well. So we need only show that $N(x, \varepsilon)$ intersects $\mathbb{R} \backslash S$.

But we're assuming that $x \notin \operatorname{int} S$, so no neighborhood $N(x, \varepsilon)$ of S can lie completely inside S, so $N(x, \varepsilon)$ must intersect $\mathbb{R} \backslash S$, and we're done. $\square$

Section 3.5:

2. Mark each statement as True or False. Justify each answer.

(a) Some unbounded sets are compact. **Solution: False.** By the Heine-Borel Theorem, compact $\Rightarrow$ bounded, so by the contrapositive, not bounded $\Rightarrow$ not compact.

(b) If S is a compact subset of $\mathbb{R}$, then there is at least one point in $\mathbb{R}$ that is an accumulation point of S. **Solution: False.** The set $S = \{3\}$ is compact, but has no accumulation points in $\mathbb{R}$.

(c) If S compact and x is an accumulation point of S, then $x \in S$. **Solution: True.** If S is compact, then it's closed, by the Heine-Borel Theorem. But a closed set contains all of its accumulation points, by Theorem 3.4.17(a).

(d) If S is unbounded, then S has at least one accumulation point. **Solution: False.** The set $\mathbb{N}$ is unbounded, but has no accumulation points.

(e) Let $\mathcal{F} = \{A_i \colon i \in \mathbb{N}\}$ and suppose that the intersection of any finite subfamily of $\mathcal{F}$ is nonempty. If $\cap \mathcal{F} = \emptyset$, then for some $k \in \mathbb{N}$, A_k is not compact. **Solution: True.** Theorem 3.5.7 tells us the following: Let $\mathcal{F} = \{A_i \colon i \in \mathbb{N}\}$ and suppose that the intersection of any finite subfamily of $\mathcal{F}$ is nonempty. If each A_i is compact, then the intersection of the A_i's is nonempty. The contrapositive of this last statement is: If $\cap \mathcal{F} = \emptyset$, then for some $k \in \mathbb{N}$, A_k is not compact. s

3. Show that each subset of $\mathbb{R}$ is not compact by describing an open cover for it that has no finite subcover. (c) $\mathbb{N}$ **Solution:** An open cover of $\mathbb{N}$ is $\mathcal{C} = \{(n - \frac{1}{4}, n + \frac{1}{4}) \colon n \in \mathbb{N}\}$ (clearly $\mathbb{N}$ is contained in the union of these sets, and clearly each of the intervals in the collection is open). To prove that this open cover has no finite subcover, let $\mathcal{B}$ be a finite collection of the intervals $(n - \frac{1}{4}, n + \frac{1}{4})$. Let n_0 be the largest of the integers n appearing; that is, $n_0 = \max\{n \colon (n - \frac{1}{4}, n + \frac{1}{4}) \in \mathcal{B}\}$. Note that $n_0 + 1$ is not in any of the intervals making up $\mathcal{B}$, since clearly, an upper bound for the union of the intervals in $\mathcal{B}$ is $n_0 + \frac{1}{4}$. So $\mathbb{N}$ is not contained in $\mathcal{B}$, so we have found an open cover $\mathcal{C}$ of $\mathbb{N}$ that has no finite subcover.

4. Prove that the intersection of any collection of compact sets is compact. **Solution:** Let $\mathcal{C}$ be a collection of compact sets. By definition of intersection,

$$\cap_{C \in \mathcal{C}} C \subseteq B,$$

where B is any one of the sets in $\mathcal{C}$. By assumption, B is bounded, and clearly any subset of a bounded set is bounded. (This follows, for example, from Exercise 8, Section 3.3.) So $\cap_{C \in \mathcal{C}} C$ is bounded.

Moreover, each element of $\mathcal{C}$ is closed, and therefore so is $\cap_{C \in \mathcal{C}} C$, by Corollary 3.4.11(a). So $\cap_{C \in \mathcal{C}} C$ is closed and bounded, and is therefore compact, by the Heine-Borel Theorem.

Section 4.1:

2. Mark each statement as true or false. Justify each answer.

(a) If $s_n \to 0$, then for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $n \geq N$ implies $s_n < \varepsilon$. **Solution: True.** The definition of limit tells us that, if $s_n \to 0$, then for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $n \geq N$ implies $|s_n| < \varepsilon$. But $|s_n| < \varepsilon \Rightarrow -\varepsilon < s_n < \varepsilon$, which certainly imples $s_n < \varepsilon$.

(b) If for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $n \geq N$ implies $s_n < \varepsilon$, then $s_n \to 0$. **Solution: False.** Consider $s_n = -3$. Since $-3 < 0$, it's certainly true that, if $\varepsilon > 0$, then for all $n \geq 1$, $s_n < \varepsilon$. But $s_n \not\to 0$.

(c) Given sequences (s_n) and (a_n), if, for some $s \in \mathbb{R}$, $k > 0$ and $m \in \mathbb{N}$ we have $|s_n - s| \leq k|a_n|$ for all $n > m$, then $\lim s_n = s$. **Solution: False.** For this to be true, we would need the extra condition $a_n \to 0$. For example, let $s_n = 5 + \frac{1}{n}$, $s = 2$, $k = 3$, and $a_n = 2n$. For all integers $n \geq 1$, we have $|s_n - s| = |5 + \frac{1}{n} - 2| = 3 + \frac{1}{n} < k|a_n| = 6n$, But $s_n \not\to 0$.

(d) If $s_n \to s$ and $s_n \to t$, then $s = t$. **Solution: True.** This is Theorem 4.1.14.

6. Using only Definition 4.1.2, prove the following.

(c) $\lim \frac{4n+1}{n+3} = 4$. **Solution:** Let $\varepsilon > 0$. [We want $|\frac{4n+1}{n+3} - 4| = \frac{11}{n+3}$ to be $< \varepsilon$ for n large enough. Solving $11/(n+3) < \varepsilon$ gives $n > (11 - 3\varepsilon)/\varepsilon$. So here's what we write.] Let $N \in \mathbb{N}$ be any integer larger than $11/\varepsilon - 3$. Then

$$n \geq N \Rightarrow \left| \frac{4n+1}{n+3} - 4 \right| = \frac{11}{n+3} < \frac{11}{(11/\varepsilon - 3) + 3} = \varepsilon.$$

So

$$\lim \frac{4n+1}{n+3} = 4.$$

(d) $\lim \frac{\sin n}{n} = 0$. **Solution:** Let $\varepsilon > 0$. [We want $|(\sin n)/n - 0| = |\sin n|/n < \varepsilon$. But $|\sin n| \leq 1$ for all n, so it will be enough to have $1/n < \varepsilon$, or $n > 1/\varepsilon$.] Let N be any integer larger than $1/\varepsilon$. Then

$$\left|\frac{\sin n}{n} - 0\right| = \frac{|\sin n|}{n} \leq \frac{1}{n} < \frac{1}{1/\varepsilon} = \varepsilon.$$

So

$$\lim \frac{\sin n}{n} = 0.$$

13. Suppose that $(a_n), (b_n)$, and (c_n) are sequences such that $a_n \leq b_n \leq c_n$ for all $n \in \mathbb{N}$ and such that $\lim a_n = \lim c_n = b$. Prove than $\lim b_n = b$. (Note: this result is sometimes called the **squeeze law.**)

Solution: Let $\varepsilon > 0$. [We want to show that $|b_n - b| < \varepsilon$ for n large enough. The idea is that both a_n and c_n will be close to b for n large enough, and since b_n is in between a_n and c_n, b has nowhere to go other than b. Here's how we make this formal.]

Since $a_n \leq b_n \leq c_n$, we have

$$a_n - b \leq b_n - b \leq c_n - b. \qquad \text{(SQ1)}$$

(I've called this equation (SQ1) to remind us of the squeeze law.)

Let $N_1 \in \mathbb{N}$ be such that $n \geq N_1 \Rightarrow |a_n - b| < \varepsilon$. Note that, since $x \leq |x|$ always, we find that, for such n, we have $a_n - b < \varepsilon$, or, multiplying by -1,

$$-\varepsilon < a_n - b. \qquad \text{(SQ2)}$$

Now let $N_2 \in \mathbb{N}$ be such that $n \geq N_2 \Rightarrow |c_n - b| < \varepsilon$. Note that, since $c_n - b \leq |c_n - b|$ always, we find that, for such n, we have

$$c_n - b < \varepsilon. \qquad \text{(SQ3)}$$

So suppose $N = \max\{N_1, N_2\}$. Then for $n \geq N$, both (SQ2) and (SQ3) are true. Putting this together with (SQ1) tells us that, for $n \geq N$,

$$-\varepsilon < a_n - b \leq b_n - b \leq c_n - b < \varepsilon,$$

which certainly implies $-\varepsilon < b_n - b < \varepsilon$, which is the same as saying $|b_n - b| < \varepsilon$. So $b_n \to b$. $\square$