Assignment:

Section 3.1 (pp. 108-113): Exercises 2, 13; **Section 3.2 (pp. 120-122):** Exercises 2, 3(a)(e)(f)(j), 5, 7 , 12; **Section 3.3 (pp. 131-134):** Exercises 1, 2, 3(a)(b)(c)(d)(f)(i)(m)(n), 4, 8, 10.

Section 3.1:

2. Mark each statement as True or False. Justify each answer.

(a) A proof using mathematical induction consists of two parts: establishing the basis for induction and verifying the induction hypothesis. **Solution: False.** A proof using mathematical induction consists of two parts: establishing the basis for induction and verifying the induction **step**.

(b) Suppose m is a natural number greater than 1. To prove $P(k)$ is true for all $k \geq m$, we must first show that $P(k)$ is false for all k such that $1 \leq k < m$. **Solution: False.** Whether $P(k)$ is true for all $k \geq m$ might have nothing to do with whether $P(k)$ is false for all k such that $1 \leq k < m$. Rather, to prove $P(k)$ is true for all $k \geq m$ (using induction), we must first show that $P(k)$ is true for $k = m$.

13. Prove that $5^{2n} - 1$ is a multiple of 8 for all $n \in \mathbb{N}$.

Solution: Let A_n be the statement "$5^{2n} - 1$ is a multiple of 8."

Is A_1 true? $5^{2 \cdot 1} - 1 = 24$, which is a multiple of 8. So A_1 is true.

Now assume A_k. So $5^{2k} - 1$ is a multiple of 8: say $5^{2k} - 1 = 8m$, for some $m \in \mathbb{Z}$. Then

$$5^{2(k+1)} - 1 = 25 \cdot 5^{2k} - 1 = 25 \cdot (5^{2k} - 1) + 25 - 1 = 25 \cdot (5^{2k} - 1) + 24 = 25 \cdot 8m + 24 = 8 \cdot (25m + 3),$$

so A_{k+1} follows.

Since A_1 is true and $A_k \Rightarrow A_{k+1}$ for all $k \in \mathbb{N}$, we find by mathematical induction that A_n is true for all $n \in \mathbb{N}$.

Section 3.2:

2. Mark each statement as True or False. Justify each answer.

(a) Axioms A1 to A5, M1 to M5, DL, and O1 to O4 describe an algebraic system known as an ordered field. **Solution: True.** It says so right in the book.

(b) If $x, y \in \mathbb{R}$ and $x < y + \varepsilon$ for every $\varepsilon > 0$, then $x < y$. **Solution: False.** $5 < 5 + \varepsilon$ for every $\varepsilon > 0$, but $5 \not< 5$.

(c) If $x, y \in \mathbb{R}$, then $|x+y| = |x|+|y|$. **Solution:** False. If $x, y \in \mathbb{R}$, then $|x+y| \leq |x|+|y|$ (this is the triangle inequality), but the two sides are not necessarily equal. For example, $|-3+5| \neq |-3| + |5|$.

3. Let x, y and z be real numbers. Prove the following.

(a) $-(-x) = x$. **Solution:** Start with the equation $x + (-x) = 0$, true by Axiom A5. By Axiom A1, we can add $-(-x)$ to both sides, to get

$$x + (-x) + (-(-x)) = 0 + (-(-x)). \tag{1}$$

But $-(-x)$ is, by definition, the thing that, when added to $-x$, gives you zero. So the left side of (1) equals $x + 0$, which equals x by Axiom A4. The right side equals $-(-x)$, again by Axiom A4. The two sides are equal, so $-(-x) = x$, as required. □

(e) If $x \neq 0$, then $x^2 > 0$. **Solution:** If $x \neq 0$ then, by Axiom O1, we have $x > 0$ or $x < 0$. We consider these two cases. (i) $x > 0$: multiply both sides by x. By Axiom O4, the direction of the inequality doesn't change, so we get $x^2 > 0 \cdot x = 0$, the last step by Theorem 3.2.2(b). (ii) $x < 0$: multiply both sides by x. By Theorem 3.2.2(g), the direction of the inequality changes, so we get $x^2 > 0 \cdot x = 0$, the last step by Theorem 3.2.2(b). So if $x \neq 0$, then $x^2 > 0$. □

(f) $0 < 1$. **Solution:** By Axiom M4, $1 \neq 0$. So by part (f) of this exercise, $1 \cdot 1 > 0$. But By Axiom M4, $1 \cdot 1 = 1$. So $1 = 1 \cdot 1 > 0$, and we're done. □

(j) If $0 < x < y$, then $0 < 1/y < 1/x$. **Solution:** Suppose $0 < x < y$. Since x and y are both nonzero, the inverses $1/x$ and $1/y$ exist, by axiom M5. Note that $x > 0 \Rightarrow 1/x > 0$ as well. Why? Well, $1/x \neq 0$ by Axiom M5. Moreover, if $1/x$ were negative, then multiplying the equation $x > 0$ (which is true by assumption) by $1/x$ would give $x \cdot (1/x) < 0$ by Theorem 3.2.2(g), or $1 < 0$, which is false by part (f) of this exercise.

Similarly, $1/y > 0$. So we can multiply both sides of the given inequality $x < y$ by $1/x$ and then by $1/y$, and the direction of the inequality doesn't change, by Axiom 04. We get

$$\frac{1}{y} \cdot \frac{1}{x} \cdot x < \frac{1}{y} \cdot \frac{1}{x} \cdot y.$$

The left hand side equals $(1/y) \cdot 1 = 1/y$, by Axioms M4 and M5. The right hand side equals $(1/x) \cdot (1/y) \cdot y$ by Axiom M2, which equals $(1/x) \cdot 1 = (1/x)$ by Axioms M4 and M5. So we have shown that $1/y < 1/x$. But we also know that $1/y > 0$, so we have $0 < 1/y < 1/x$.

12. Let $S = \{a, b\}$ and define two operations $\oplus$ and $\otimes$ on S by the following charts:

$\oplus$	a	b
a	a	b
b	b	a

$\otimes$	a	b
a	a	a
b	a	b

(a) Verify that S together with $\oplus$ and $\otimes$ satisfies the axioms of a field.

Solution: This is a long problem, because we need to show that Axioms A1–A4, M1–M4, and DL are satisfied. We'll leave out some parts of this, but will get across the main ideas.

A1, M1: This just says that $\oplus$ and $\otimes$ are closed and well-defined. The tables show they're closed, because the result of the $\oplus$ or $\otimes$ of any two elements of S is an element of S. That $\oplus$ and $\otimes$ are well-defined follows from the fact that they're defined by the tables!

A2, M2: Note that showing either of these operations is commutative amounts to showing that the entry in the ith row and jth column of either table equals the entry in the jth row and ith column. There are only eight things to check (four for each table), and this can be done case-by-case. (E.g. from the $\otimes$ table, $a \otimes b = a$ and $b \otimes a = a$, so $a \otimes b = b \otimes a$.)

A3, M3: Let's start with $\oplus$. We need to show that $\forall x, y, z \in S, x \oplus (y \oplus z) = (x \oplus y) \oplus z$. Note that there are eight choices for the quantity on the left: two choices for x, two for y, and two for z. Similarly on the right. For example, from the table, $a \oplus (b \oplus a) = a \oplus b = b$, while $(a \oplus b) \oplus a = b \oplus a = b$. So $a \oplus (b \oplus a) = (a \oplus b) \oplus a$. And so on down the line. And similarly for M3.

A4,M4: We check directly that $a \oplus a = a$ and $b \oplus a = b$, and that no other element y of S satisfies $x \oplus y = x$ for all $x \in S$. So our "0" element is a. Similarly, we check that our "1" element is b.

A5, M5: $a \oplus a = a = 0$ and $b + b = a = 0$, so $a = -a$ and $b = -b$. Similarly, $b \otimes b = b = 1$, so $b = 1/b$. (We don't need to check for $1/a$, since $a = 0$, so it has no "reciprocal.")

DL: There are eight ways of constructing an expression $x \otimes (y \oplus z)$: two choices for x, two for y, and two for z. Actually there's some redundancy here, since we have already seen that $\oplus$ is commutative, so we already know, for example, that $a \otimes (b \oplus a) = a \otimes (a \oplus b)$. So there are only four choices: $a \otimes (a \oplus a)$, $a \otimes (a \oplus b)$, $b \otimes (a \oplus a)$, and $b \otimes (a \oplus b)$. Just check that each one equals the corresponding quantity $(x \otimes y) \oplus z$, and you're done. □

(b) Identify the elements of S that are "0," "1," and "−1." **Solution:** We already identified that $a = 0$ and $b = 1$ above. By definition, -1 is what you add to 1 (that is, to b) to get 0 (that is, to get a) so by the first table, $-1 = b$.

Section 3.3:

1. Mark each statement as true or false. Justify each answer.

(a) If a nonempty subset of $\mathbb{R}$ has an upper bound, then it has a least upper bound. **Solution: True.** This is the completeness axiom.

(b) If a nonempty subset of $\mathbb{R}$ has an infemum, then it is bounded. **Solution: False.** $(0, \infty)$ has an infemum but is not bounded.

(c) Every nonempty bounded subset of $\mathbb{R}$ has a maximum and a minimum. **Solution: False.** $(0, 1)$ has no minimum or maximum.

(d) If m is an upper bound of S and $m' < m$, then m' is not an upper bound of S. **Solution: False.** This would be true with "least upper bound" in place of "upper bound," but is false as is. E.g. 5 is an upper bound for $[0, 3)$ and $4 < 5$, but 4 is an upper bound for $[0, 3)$ as well.

(e) If $m = \inf S$ and $m' < m$, then m' is a lower bound of S. **Solution: True.** To say that $m = \inf S$ is to say that $m \leq x$ for every $x \in S$. If this is the case, and $m' < m$, then $m' \leq x$ for every $x \in S$ as well, so m' is also a lower bound for S.

(f) For each real number x and each $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that $n\varepsilon > x$. **Solution: True.** This is Theorem 3.3.10, with x replaced by ε and y replaced by x.

8. Let S and T be nonempty bounded subsets of $\mathbb{R}$ with $S \subseteq T$. Prove that $\inf T \leq \inf S \leq \sup S \leq \sup T$.

Solution: We first show that $\inf T \leq \inf S$: if $x \in S$, then $x \in T$ so, by definition of inf, we have $\inf T \leq x$. This shows that $\inf T$ is $\leq$ every element of S, so $\inf T$ is a lower bound for S, and is therefore no larger than the greatest lower bound for S, so $\inf T \leq \inf S$.

We next show that $\inf S \leq \sup S$: let $x \in S$. Then, by definition of upper and lower bounds, $\inf S \leq x \leq \sup S$. So $\inf S \leq \sup S$.

Finally, we show that $\sup S \leq \sup T$: if $x \in S$, then $x \in T$ so, by definition of sup, we have $\sup T \geq x$. This shows that $\sup T$ is $\geq$ every element of S, so $\sup T$ is an upper bound for S, and is therefore no smaller than the least upper bound for S, so $\sup S \leq \sup T$.

10. (a) Prove: if x and y are real numbers with $x < y$, then there are infinitely many rational numbers in the interval $[x, y]$.

Solution: Let A_n be the statement "if x and y are real numbers with $x < y$, then there are n rational numbers in the interval $[x, y]$." We prove by induction that A_n is true for all $n \in \mathbb{N}$. First: A_1 is true by Theorem 3.3.13. Next: suppose A_k is true. That is, for real numbers x and y with $x < y$, there are k rational numbers $x_1, x_2, \ldots, x_k$ in the interval $[x, y]$. Assume these k numbers are arranged in increasing order. By Theorem 3.3.13, there is a rational number x_0 with $x_1 < x_0 < x_2$. This gives us $k + 1$ rational numbers in $[x, y]$.

So A_1 is true and A_k implies A_{k+1}. So A_n is true for all n, meaning $[x, y]$ contains infinitely many rational numbers, for all $x, y \in \mathbb{R}$ with $x < y$.

(b) Repeat part (a) for irrational numbers.

Solution: Similar to part (a), but use Theorem 3.3.15.