Assignment:

S-POP Part B(ii): Exercises B(ii)-2,4,7; **S-POP Part B(iii):** Exercises B(iv)-2, 4, 5; **Text Section 2.3 (pp. 77–82):** Exercises 2, 4a, 9, 12, 13, 27.

S-POP Part B(ii):

4. Show that, for any sets A, B, and C, we have $A \cap B \subseteq A \cup C$.

Solution: Let $x \in A \cap B$. Then $x \in A$ and $x \in B$ by definition of intersection, so certainly $x \in A$. But then $x \in A \cup C$ by definition of union.

So $A \cap B \subseteq A \cup C$. □

7. Using the strategy of Proposition B(ii)-2, prove that, for any sets X, Y, and Z,

$$X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z).$$

Solution: Though there are many ways to prove this, you are asked in this exercise to prove it "Using the strategy of Proposition B(ii)-2." This means: show that any x in the set on the left is contained in the set on the right, and vice versa.

So let's do that: let $X \cup (Y \cap Z)$. Then either $x \in X$ or $x \in Y \cap Z$. In the first case ($x \in X$), we have $x \in X \cup Y$ and $x \in X \cup Z$ by definition of union, so $x \in (X \cup Y) \cap (X \cup Z)$ by definition of intersection. In the second case ($x \in Y \cap Z$), we have $x \in Y$ and $x \in Z$ by definition of intersection, so $x \in X \cup Y$ and $x \in X \cup Z$ by definition of union, so again, $x \in (X \cup Y) \cap (X \cup Z)$ by definition of intersection. So in either case, $x \in (X \cup Y) \cap (X \cup Z)$. So $X \cup (Y \cap Z) \subseteq (X \cup Y) \cap (X \cup Z)$.

Now let $x \in (X \cup Y) \cap (X \cup Z)$. Then $x \in X \cup Y$ and $x \in X \cup Z$, by definition of intersection. We consider two cases: $x \in X$ and $x \notin X$. In the first case ($x \in X$), we have $X \cup (Y \cap Z)$ by definition of intersection. In the second case ($x \notin X$), the fact that $x \in X \cup Y$ and $x \in X \cup Z$ implies $x \in Y$ and $x \in Z$, by definition of union. But then $x \in Y \cap Z$ by definition of intersection, so $x \in X \cup (Y \cap Z)$ by definition of union. So in either case, $x \in X \cup (Y \cap Z)$. So $(X \cup Y) \cap (X \cup Z) \subseteq X \cup (Y \cap Z)$.

Since $X \cup (Y \cap Z) \subseteq (X \cup Y) \cap (X \cup Z)$ and $(X \cup Y) \cap (X \cup Z) \subseteq X \cup (Y \cap Z)$, we conclude that $X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z)$. □

S-POP Part B(iii):

4. Prove that $\exists k \in \mathbb{Z}$ such that k can be expressed as a sum of two squares in two different ways. Hint: you don't have to go too far; there's a $k < 100$ that works.

Solution: $50 = 25 + 25 = 5^2 + 5^2 = 49 + 1 = 7^2 + 1^2$.

5. **(a)** Prove that:

$$\forall x \in \mathbb{R}, \exists y \in \mathbb{R} \colon x > y.$$

(b) Prove that the statement

$$\exists y \in \mathbb{R} \colon \forall x \in \mathbb{R}, x > y$$

is false.

Solution: **(a)**: given $x \in \mathbb{R}$, let $y = x - 1$. Then $x > y$. So $\forall x \in \mathbb{R}, \exists y \in \mathbb{R} \colon x > y$.

(b): Suppose $\exists y \in \mathbb{R}$ that is larger than any real number x. Then in particular $y > y$, which is false.

Text Section 2.3 (pp. 77–82):

2. Mark each statement as true or false. Justify your answer.

(a) If $f : A \to B$ and C is a nonempty subset of A, then $f(C)$ is a nonempty subset of B. **Solution: TRUE:** if $x \in A$ then $f(x) \in f(C)$, so $f(C)$ is nonempty.

(b) If $f : A \to B$ is surjective and $y \in B$, then $f^{-1}(y) \in A$. **Solution: FALSE**, though this is a bit tricky. Really we should say $f^{-1}(\{y\}) \subset A$, since technically, the inverse image of a set is a set, not a point. On the other hand, if f is also *injective*, then $f^{-1}(y)$ is a single point: namely, it's the image of y under the inverse function f^{-1}.

(c) If $f : A \to B$ and D is a nonempty subset of B, then $f^{-1}(D)$ is a nonempty subset of A. **Solution: FALSE.** This need not hold if f is not surjective. For example, consider the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$. Then $[-2, -1]$ is a nonempty subset of $\mathbb{R}$, but $f^{-1}(D)$ is empty, since nothing maps to negative numbers under f.

(d) The composition of two surjective functions is always surjective. **Solution: TRUE.** See Theorem 2.3.20(a), p. 72.

(e) If $f : A \to B$ is bijective, then $f^{-1} : B \to A$ is bijective. **Solution: TRUE.** See Theorem 2.3.24(a), p. 74.

(f) The identity function maps $\mathbb{R}$ onto $\{1\}$. **Solution: FALSE**. The identify function f is defined by $f(x) = x$. This maps $\mathbb{R}$ onto $\mathbb{R}$.

10. In each part, find a function $f : \mathbb{N} \to \mathbb{N}$ that has the desired properties.

(a) surjective, but not injective. **Solution:** $f(n) = \begin{cases} n/2 & \text{if } n \text{ is even,} \\ (n+1)/2 & \text{if } n \text{ is odd.} \end{cases}$

(E.g. $f(3) = f(4) = 2$.)

(b) injective, but not surjective. **Solution:** $f(n) = 2n$.

(c) neither surjective nor injective **Solution:** $f(n) = 1$.

(d) bijective **Solution:** $f(n) = n$.

12. Given $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$, we define the sum $f+g$ by $(f+g)(x) = f(x)+g(x)$ and the product fg by $(fg)(x) = f(x) \cdot g(x)$ for all $x \in \mathbb{R}$. Find counterexamples for the following.

(a) If f and g are bijective, then the sum $f + g$is bijective. **Solution:** $f(x) = x$ and $g(x) = -x$.

(b) If f and g are bijective, then the product fg is bijective. **Solution:** $f(x) = x$ and $g(x) = -x$.

13. Consider the function $f : A \to B$ illustrated in Figure 11.

(a) Find $f(S)$, where $S = \{2, 3, 4, 5\}$. **Solution:** $f(S) = \{b, c, e\}$.

(b) Find $f^{-1}(T)$, where $T = \{a, b, d\}$. **Solution:** $f^{-1}(T) = \{1, 2\}$.