BRIEF intro to Fourier analysis.
GOAL: to investigate Fourier's 1807 claim:
"there is no function... which cannot be expressed by a trigonometric series."

Part I: complex numbers.
Throughout, x, x_1, x_2, y, u, v denote arbitrary real numbers.

Definition 1.
(a) Let i denote a square root of -1, so $i^2 = 1$.
A complex number is a quantity $x+iy$. The set of all complex numbers is denoted $\mathbb{C}$.
Geometrically, we think of $\mathbb{C}$ as the usual xy-plane $\mathbb{R}^2$: that is, we identify $x+iy \in \mathbb{C}$ with the point $(x,y) \in \mathbb{R}^2$. In particular, since $i = 0+1i$, we identify i with $(0,1)$.

(b) Let $z = x+iy \in \mathbb{C}$. Then:

- We call x the real part of z, denoted $\operatorname{Re} z$, and call y the imaginary part of z, denoted $\operatorname{Im} z$.

- The modulus $|z|$ is the distance from z to $(0,0)$. So $|z| = \sqrt{x^2+y^2}$.
Also, the angle that z makes with the positive x-axis is called the argument of z, denoted $\operatorname{Arg} z$. We assume $\operatorname{Arg} z \in (-\pi, \pi]$.
Finally, we denote by $\bar{z}$ the reflection of z about the x-axis. So $\bar{z} = x - iy$. We call $\bar{z}$ the complex conjugate of z.

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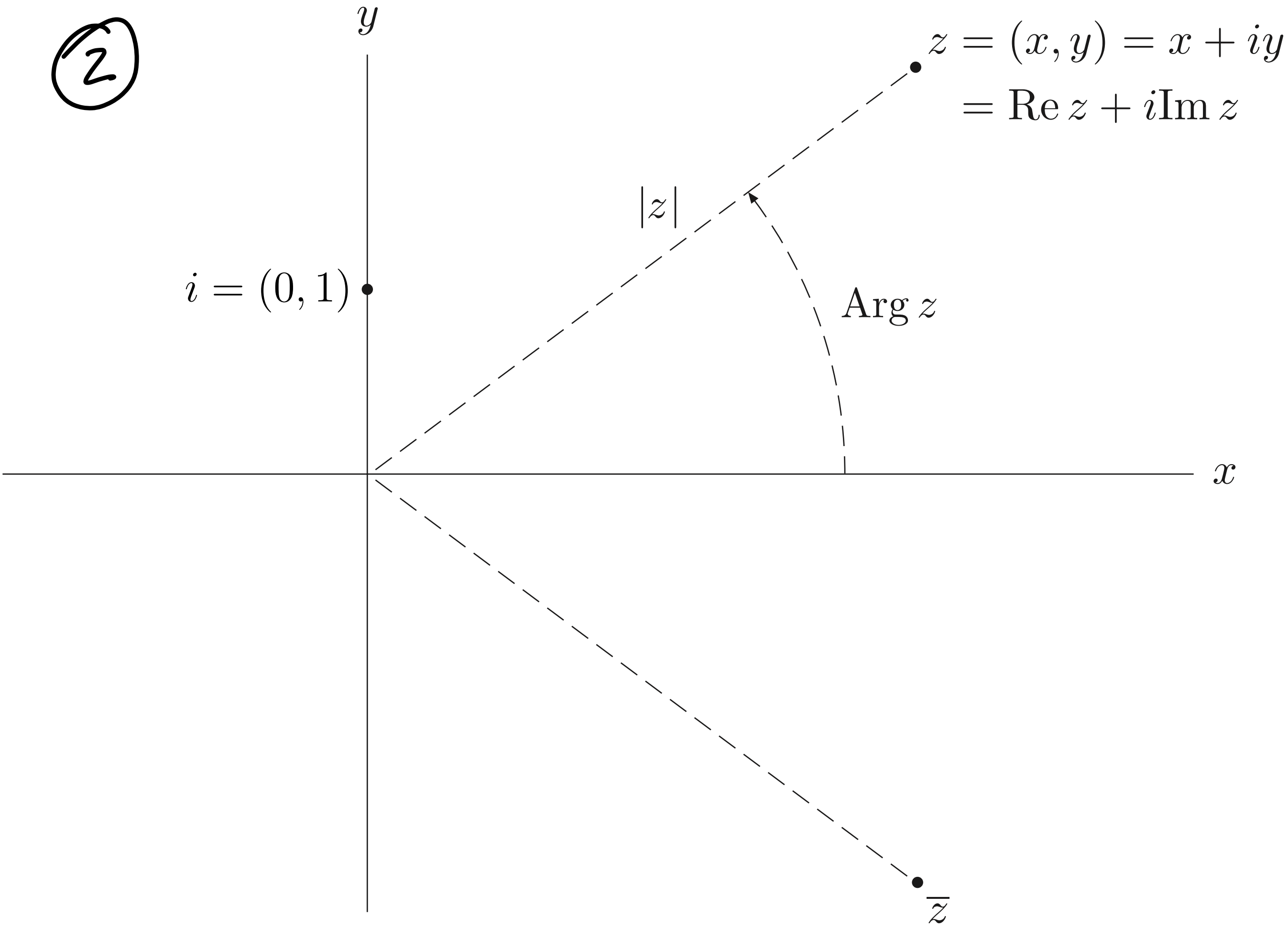


y
$z = (x, y) = x + iy$
$= \mathrm{Re}\, z + i \mathrm{Im}\, z$
$|z|$
$i = (0, 1)$
$\mathrm{Arg}\, z$
x
$\overline{z}$

(c) We define addition and multiplication in $\mathbb{C}$ by way of the operations on $\mathbb{R}$, keeping in mind that $i^2=-1$:

$$(x+iy)+(u+iv) = (x+u)+i(y+v);$$
$$(x+iy)(u+iv) = xu + x(iv) + iyu + (iy)(iv)$$
$$= (xu-yv) + i(xv+yu).$$

Exercise 1: Using the above definitions, show that

(a) $z+\bar{z} = 2\operatorname{Re} z$; $\quad z-\bar{z} = 2i \operatorname{Im} z$;

(b) $z\bar{z} = |z|^2$;

(c) If $z \neq 0$ then the unique complex number z^{-1} (also written $1/z$) such that $zz^{-1}=z^{-1}z=1$ is given by $z^{-1} = (x-iy)/(x^2+y^2)$.

Part II: the function e^{ix}.

Definition: if $x \in \mathbb{R}$, we define the complex exponential function e^{ix} by

$$e^{ix} = \cos x + i \sin x.$$

Theorem 1: properties of e^{ix}.

We have:

(a) $|e^{ix}| = 1$.

(b) $e^{ix_1}e^{ix_2} = e^{i(x_1+x_2)}$

(c) $1/e^{ix} = e^{-ix}$ (where e^{-ix} means $e^{i(-x)}$).

(d) $(e^{ix})^n = e^{inx}$ for $n \in \mathbb{N}$.

(e) $e^{in\pi} = (-1)^n$ for $n \in \mathbb{N}$.

Proof:

(a) $|e^{ix}| = |\cos x + i \sin x| = \sqrt{\cos^2 x + \sin^2 x} = 1$.

(b) $e^{ix_1}e^{ix_2} = (\cos x_1 + i \sin x_1)(\cos x_2 + i \sin x_2)$

$$= \cos x_1 \cos x_2 + i \sin x_1 \cos x_2 + i \cos x_1 \sin x_2 + i^2 \sin x_1 \sin x_2$$

$$= (\cos x_1 \cos x_2 - \sin x_1 \sin x_2) + i(\sin x_1 \cos x_2 + \cos x_1 \sin x_2)$$

trig identities

$$= \cos(x_1+x_2) + i \sin(x_1+x_2)$$

$$= e^{i(x_1+x_2)}.$$

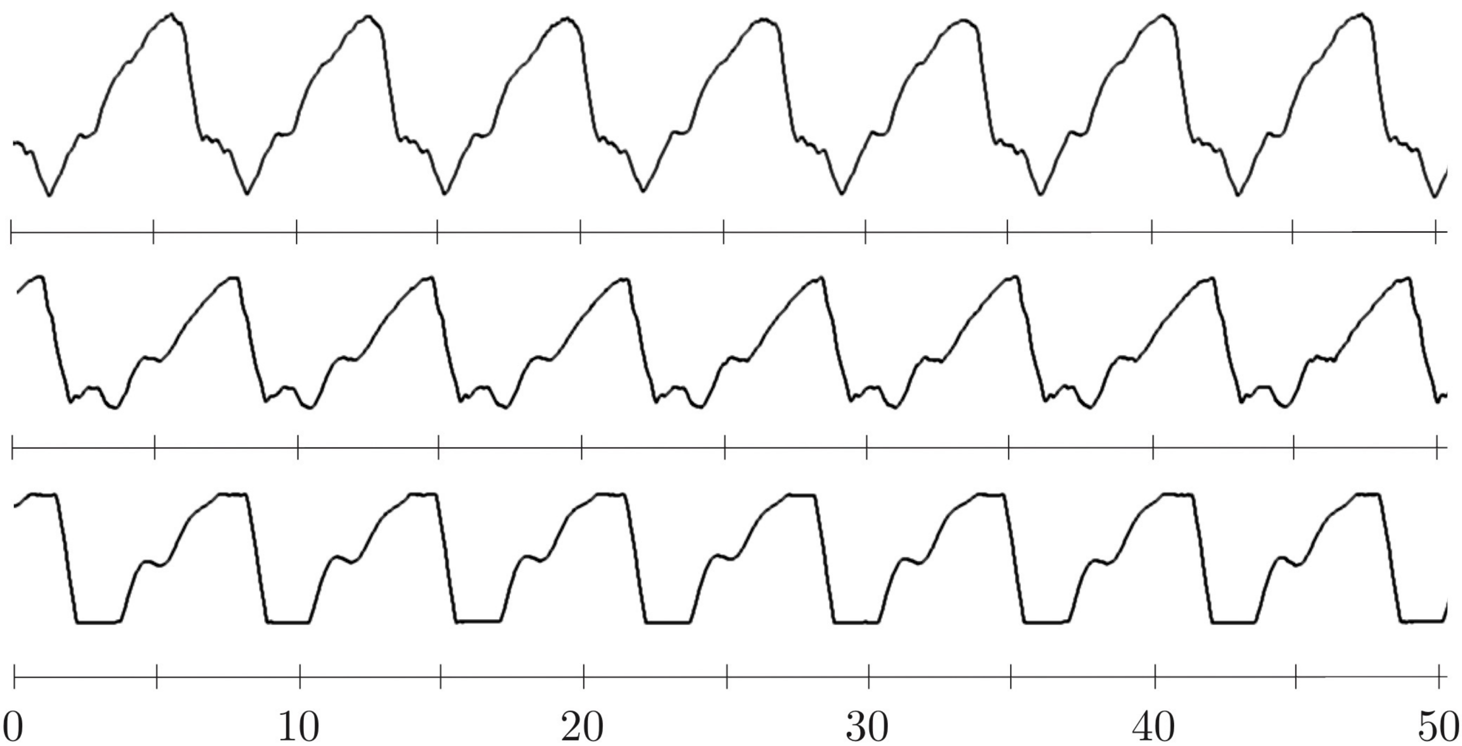
Parts (c)(d)(e): this is Exercise 2. $\square$

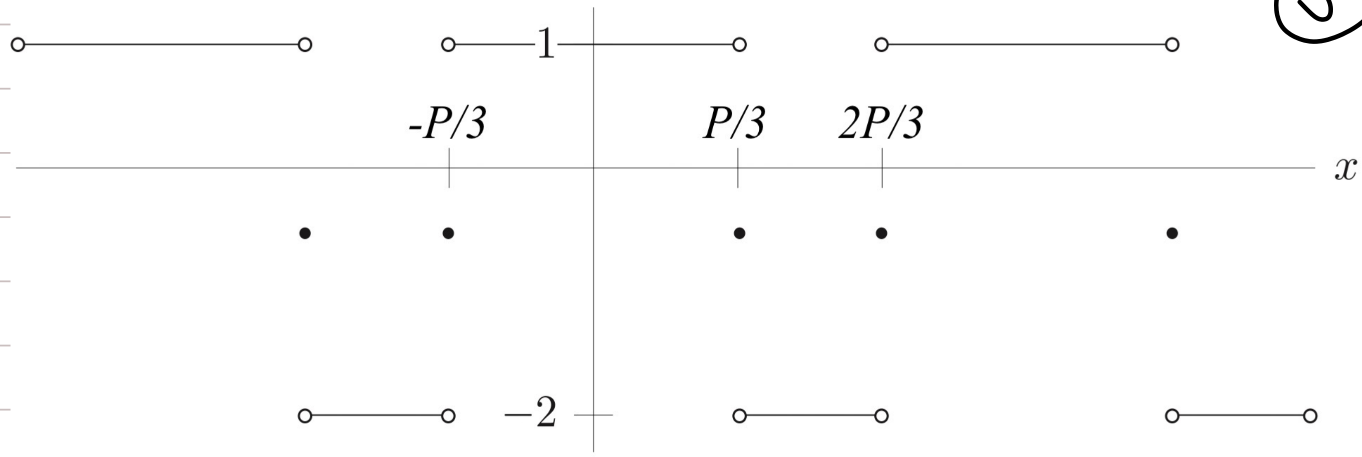
Part III: periodic functions.

Definition 2. Let $P>0$.

A function $f:\mathbb{R}\to\mathbb{R}$ is said to be *P-periodic* if $f(x+P) = f(x) \ \forall x \in \mathbb{R}$.

A P-periodic function is one whose graph repeats itself every P units.

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-P/3
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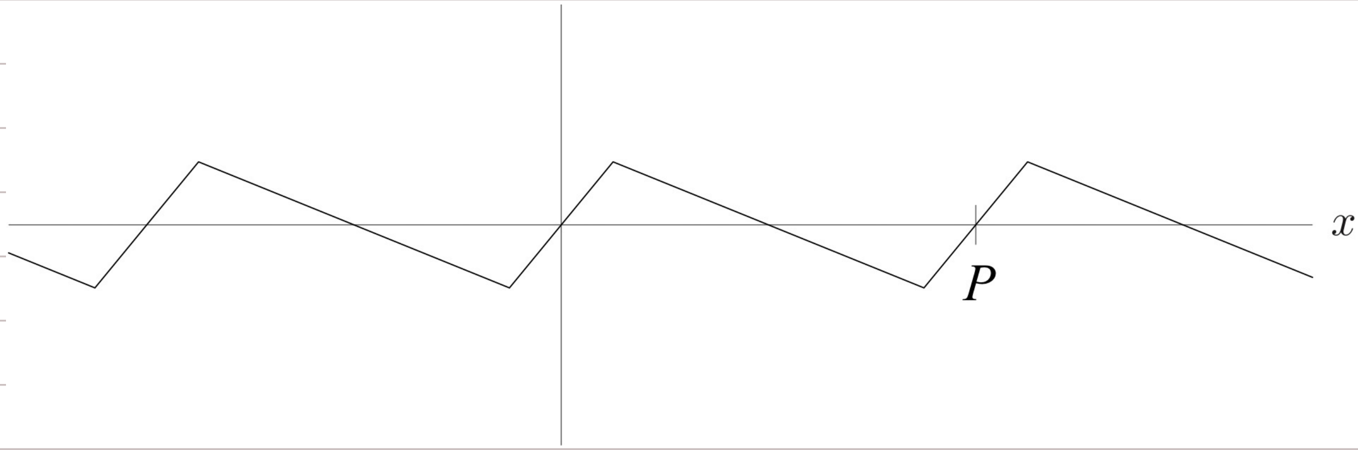
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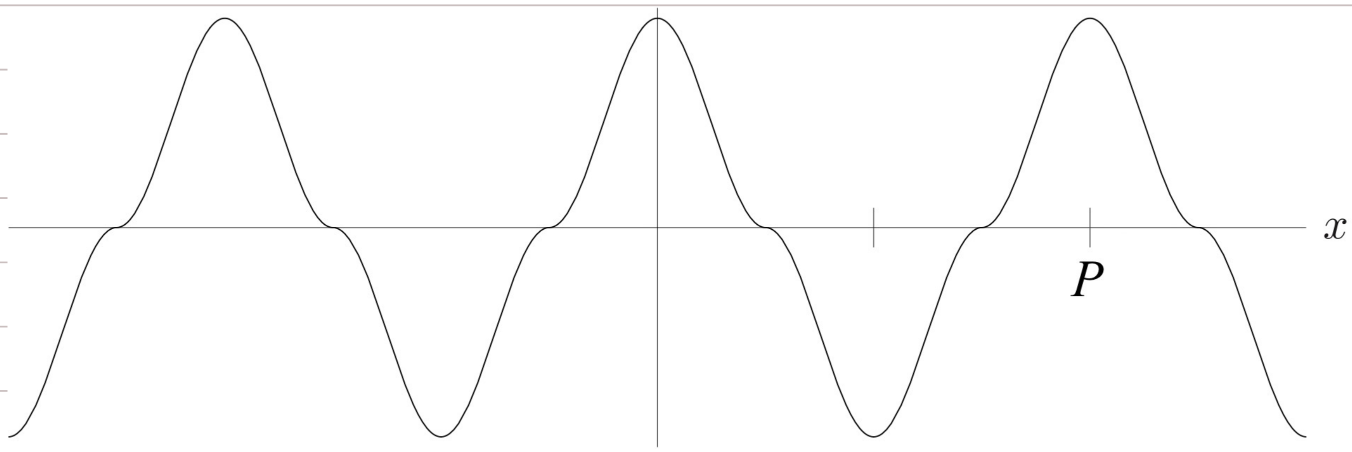
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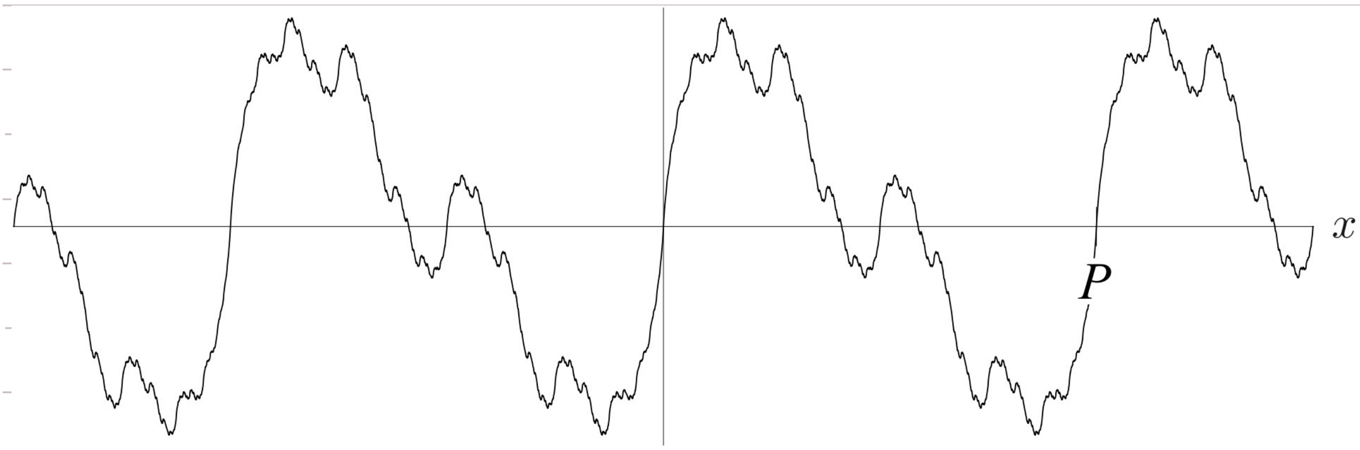
x
P

For simplicity, let's focus on 2π-periodic functions. For example, let $n \in \mathbb{Z}$. Then the function p_n defined by $p_n(x) = \cos(nx)$ is 2π-periodic, since

$$p_n(x+2\pi) = \cos(n(x+2\pi)) = \cos(nx+2\pi n) = \cos(nx) = p_n(x).$$

Similarly, $q_n(x) = \sin(nx)$ and $e_n(x) = e^{inx} = \cos(nx) + i\sin(nx)$ define 2π-periodic functions.

Now let f be 2π-periodic. If Fourier was right, and f <u>has</u> a trigonometric series, then it stands to reason that the trigonometric <u>functions</u> in that series are 2π-periodic too. So, <u>maybe</u>, f has an expression of the form

$$f(x) = \sum_{n\in\mathbb{Z}} c_n(f)\, e_n(x) = \sum_{n\in\mathbb{Z}} c_n(f)\, e^{inx}$$
$$= \sum_{n\in\mathbb{Z}} c_n(f)\left[\cos(nx) + i\sin(nx)\right], \qquad (*)$$

for appropriate numbers $c_n(f) \in \mathbb{R}$ (or $\mathbb{C}$).

More on $(*)$ next time.

SOLUTIONS TO THE ABOVE EXERCISES

Exercise 1. For these exercises, given $z \in \mathbb{C}$, write $z = x+iy$, with $x, y \in \mathbb{R}$.
Show that:

(a) $z + \bar{z} = 2\operatorname{Re} z$; $\quad z - \bar{z} = 2i \operatorname{Im} z$.

SOLUTION:

$$z + \bar{z} = (x+iy) + (x-iy) = 2x = 2\operatorname{Re} z.$$
$$z - \bar{z} = (x+iy) - (x-iy) = 2iy = 2i\operatorname{Im} z.$$

(b) $z\bar{z} = |z|^2$.

SOLUTION:

$$z\bar{z} = (x+iy)(x-iy) = x^2 - i^2y^2 = x^2+y^2 = \left(\sqrt{x^2+y^2}\right)^2$$
$$= |z|^2.$$

(c) If $z \neq 0$ then the unique complex number z^{-1} (also written $1/z$) such that $zz^{-1} = z^{-1}z = 1$ is given by $z^{-1} = (x-iy)/(x^2+y^2)$.

SOLUTION:

we check that

$$zz^{-1} = (x+iy) \cdot \frac{(x-iy)}{x^2+y^2} = \frac{x^2+y^2}{x^2+y^2} = 1.$$

Similarly, $z^{-1}z = 1$.
To see that z^{-1} is unique, suppose q is some other number with

$$zq = 1. \qquad (*)$$

Multiply both sides of $(*)$ by z^{-1} to get

$$z^{-1}zq = z^{-1}$$

or, since $z^{-1}z = 1$,

$$q = z^{-1}.$$

Exercise 2.

Prove the following parts of Theorem 1 above:

(c) $1/e^{ix} = e^{-ix}$ (where e^{-ix} means $e^{i(-x)}$).

SOLUTION:

We need to show that $e^{ix}e^{-ix} = 1$. But by part (b) of this theorem,

$$e^{ix}e^{-ix} = e^{i(x-x)} = e^{i0} = \cos 0 + i\sin 0 = 1.$$

(d) $(e^{ix})^n = e^{inx}$ for $n \in \mathbb{N}$.

SOLUTION:

We prove this by induction. Certainly it's true for $n=1$, since $(e^{ix})^1 = e^{ix} = e^{i\cdot 1\cdot x}$. Now assume $A(k)$: $(e^{ix})^k = e^{ikx}$ is true. To prove $A(k+1)$, we note that

$$\begin{aligned}(e^{ix})^{k+1} &= (e^{ix})^k e^{ix} \\ &= e^{ixk}e^{ix} && \text{(by the induction hypothesis)} \\ &= e^{i(k+1)x} && \text{(by part (b) of this theorem).}\end{aligned}$$

So $A(k+1)$ is true.

So $A(1)$ is true and $A(k) \Rightarrow A(k+1)$, so by mathematical induction, $A(n)$ is true $\forall n \in \mathbb{N}$.

(e) $e^{in\pi} = (-1)^n$ for $n \in \mathbb{N}$.

SOLUTION:

By part (d) above, with $x = \pi$,

$$\begin{aligned}e^{in\pi} &= (e^{i\pi})^n = (\cos\pi + i\sin\pi)^n \\ &= (-1+i0)^n = (-1)^n.\end{aligned}$$

Fourier series, concluded.

The story so far:

If f is a "reasonable" 2π-periodic function, then it's "reasonable" to expect that f has a trig series in terms of the 2π-periodic trig functions

$$e_n(x) = e^{inx} = \cos(nx) + i\sin(nx),$$

where $i = -1$ and $n \in \mathbb{Z}$.

That is, we might expect that

$$f(x) = \sum_{n\in\mathbb{Z}} c_n(f) e^{inx} \qquad (*)$$

for appropriate "Fourier coefficients" $c_n(f)$.

Two BIG questions:

(1) Is this assumption correct, and if so, how "reasonable" does f have to be for $(*)$ to hold?

(2) If $(*)$ DOES hold, what IS $c_n(f)$??

It's easiest to answer (2) first, like this. Suppose $(*)$ is true. Let's multiply both sides of $(*)$ by e^{-ikx}, where $k \in \mathbb{Z}$, then integrate both sides over $[-\pi, \pi]$, then divide both sides by 2π. We get:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-ikx}\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\sum_{n\in\mathbb{Z}} c_n(f) e^{inx}\right) e^{-ikx}\,dx$$

$$= \frac{1}{2\pi}\sum_{n\in\mathbb{Z}} c_n(f)\int_{-\pi}^{\pi} e^{i(n-k)x}\,dx. \qquad (*')$$

pull stuff that doesn't depend on x outside the integral

NOW:

(a) If $n=k$, then $e^{i(n-k)x} = e^{i0x} = e^{i0} = \cos 0 + i\sin 0 = 1$, so the integral on the right side of (*') equals
$$\int_{-\pi}^{\pi} 1\,dx = 2\pi.$$

(b) If $n \neq k$, then the integral on the right side of (*') equals
$$\int_{-\pi}^{\pi} e^{-i(n-k)x}\,dx = \left.\frac{e^{-i(n-k)x}}{-i(n-k)}\right|_{-\pi}^{\pi}$$

antidifferentiate like i is any other constant

$$= \frac{e^{-i(n-k)\pi} - e^{-i(n-k)(-\pi)}}{-i(n-k)} = \frac{(-1)^{k-n} - (-1)^{n-k}}{-i(n-k)} = 0 \;!!$$

Exercise 2d, last time

So (*') gives
$$\frac{1}{2\pi i}\int_{-\pi}^{\pi} f(x)e^{-ikx}\,dx = \frac{1}{2\pi}\cdot c_k(f)\cdot 2\pi = c_k(f).$$

That is, replacing k by n,
$$c_n(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx.$$

CONCLUSION:

<u>If</u> f is 2π-periodic and <u>has</u> a Fourier series (*), <u>then</u>
$$c_n(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-ikx}\,dx. \qquad (**)$$

<u>Exercise 1.</u> Let f be the 2π-periodic function defined on $[-\pi,\pi]$ by $f(x)=x^2$.

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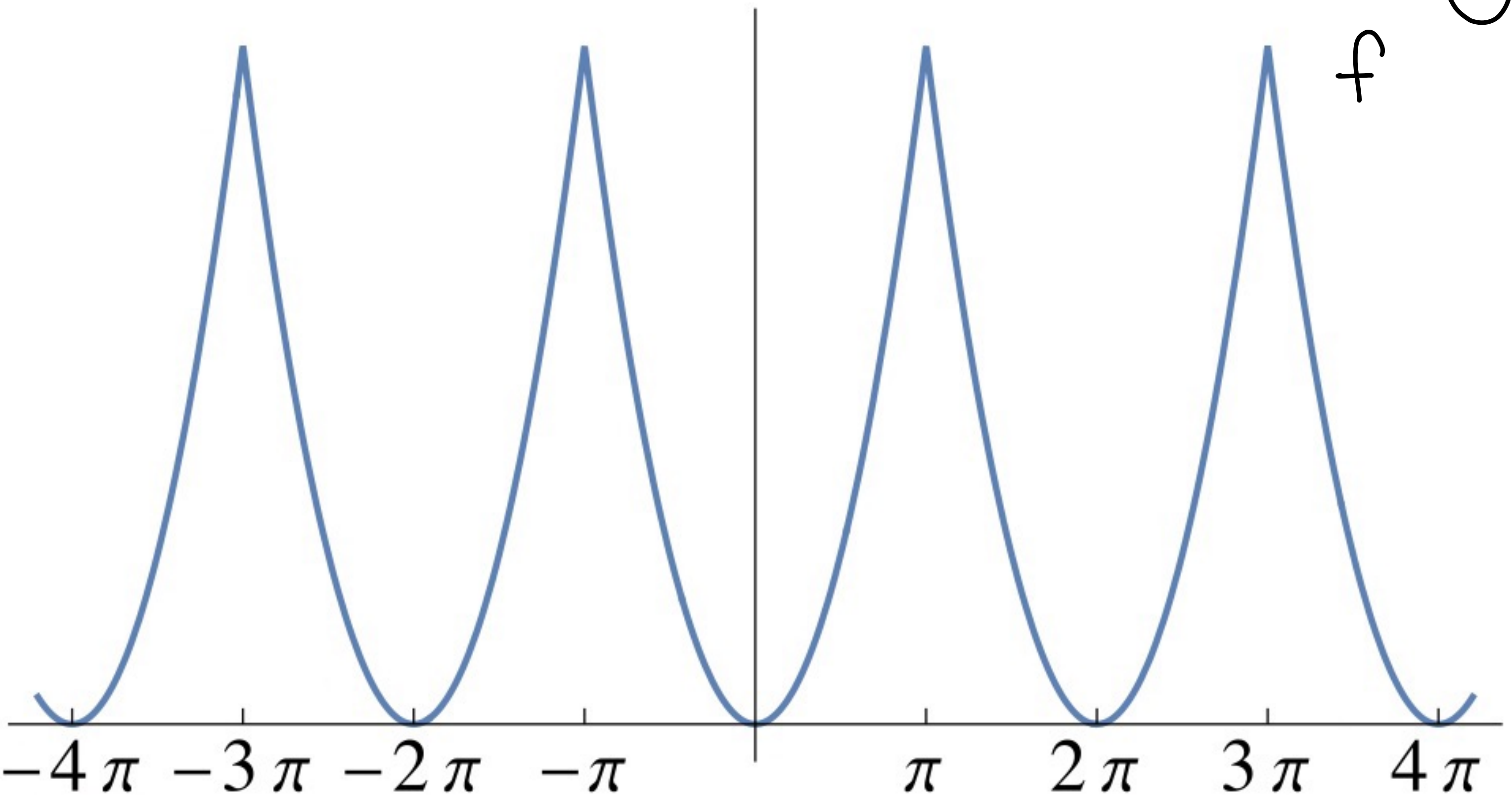

Show that

$$c_n(f) = \begin{cases} \pi^2/3 & \text{if } n=0, \\ 4\pi(-1)^n/n^2 & \text{if } n \neq 0. \end{cases}$$

Partial solution.

We have, by (**),

$$c_n(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-inx}\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2 e^{-inx}\,dx.$$

Now we use (without proof) the formula

$$\int x^2 e^{ax}\,dx = \begin{cases} x^3/3 + C & \text{if } a=0, \\ \dfrac{e^{ax}(a^2x^2 - 2ax - 2)}{a^3} + C & \text{if } a \neq 0 \end{cases}$$

(where a is any constant $\neq 0$) to finish.

FINALLY, we address, without proof, question (1): WHEN (for which f) does (*) hold?

We have the following, due to Dirichlet et. al.!

Theorem 1.

If f is 2π-periodic and continuous on $\mathbb{R}$, and f' is piecewise continuous on $\mathbb{R}$, then

$$f(x) = \sum_{n\in\mathbb{Z}} c_n(f) e^{inx} \qquad \forall x \in \mathbb{R}.$$

Exercise 2.

Use Exercise 1 and the above theorem to show that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Hint: use Theorem 1 with $x = \pi$.

SOLUTIONS TO THE ABOVE EXERCISES

Exercise 1. Let f be the 2π-periodic function defined on $[-\pi,\pi]$ by $f(x)=x^2$.
Show that
$$c_n(f)=\begin{cases}2\pi^2/3 & \text{if } n=0,\\ 4\pi(-1)^n/n^2 & \text{if } n\neq 0.\end{cases}$$

SOLUTION:

We have, by

$$c_n(f)=\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx=\frac{1}{2\pi}\int_{-\pi}^{\pi} x^2 e^{-inx}\,dx.$$

Now we use the formula

$$\int x^2 e^{ax}\,dx=\begin{cases}x^3/3+C & \text{if } a=0,\\ \dfrac{e^{ax}(a^2x^2-2ax-2)}{a^3}+C & \text{if } a\neq 0.\end{cases}$$

It gives us:
$$c_0(f)=\frac{x^3}{3}\Big|_{-\pi}^{\pi}=\frac{\pi^3-(-\pi)^3}{3}=\frac{2\pi^3}{3}.$$

Also, for $n\neq 0$,
$$c_n(f)=\frac{e^{-inx}((-in)^2x^2-2(-in)x-2)}{(-in)^3}\Bigg|_{-\pi}^{\pi}$$
$$=\frac{1}{(-in)^3}\left[\begin{array}{l}e^{-in\pi}(-n^2\pi^2+2in\pi-2)\\ -e^{in\pi}(-n^2\pi^2-2in\pi-2)\end{array}\right]$$

$$=\frac{1}{(-in)^3}\left[(-1)^n(-n^2\pi^2+2in\pi-2)-(-1)^n(-n^2\pi^2-2in\pi-2)\right]$$

$$=\frac{1}{(-in)^3}\cdot(-1)^n\cdot 4in\pi=\frac{4\pi(-1)^n}{n^3}.$$

Exercise 2.

Use Exercise 1 and the above theorem to show that
$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Hint: use Theorem 1 with $x=\pi$.

SOLUTION:

By Exercise 1 and the above theorem, we have
$$f(x) = \sum_{n\in\mathbb{Z}} c_n(f) e^{inx} = c_0(f) e^{i0x} + \sum_{n\neq 0} c_n(f) e^{inx}$$
$$= \frac{4\pi^2}{3} + 4\pi \sum_{n\neq 0} \frac{(-1)^n}{n^2} e^{inx}.$$

Plug in $x=\pi$. Since $f(x)=x^2$ on $[-\pi,\pi]$, we have $f(\pi)=\pi^2$, so we get
$$\pi^2 = \frac{2\pi^2}{3} + 4\pi \sum_{n\neq 0} \frac{(-1)^n}{n^2} e^{in\pi} = \frac{2\pi^2}{3} + \sum_{n\neq 0} \frac{(-1)^n \cdot (-1)^n}{n^2}$$
$$= \frac{2\pi^2}{3} + \sum_{n\neq 0} \frac{1}{n^2}, \qquad (*')$$

since $(-1)^n \cdot (-1)^n = (-1)^{2n} = 1$. We solve $(*')$ for the infinite series, to get
$$\sum_{n\neq 0} \frac{1}{n^2} = \pi^2 - \frac{2\pi^2}{3} = \frac{\pi^2}{3}.$$

But now note that the series on the left is just twice the series you get by summing over positive n, since $1/(-n)^2 = 1/n^2$. So
$$2\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{3}, \quad \text{or} \quad \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$