

BRIEF intro to Fourier analysis.

GOAL: to investigate Fourier's 1807 claim:

"there is no function... which cannot be expressed by a trigonometric series."

Part I: complex numbers.

Throughout, x, x_1, x_2, y, u, v denote arbitrary real numbers.

Definition 1.

(a) Let i denote a square root of -1 , so $i^2 = -1$.

A complex number is a quantity $x+iy$. The set of all complex numbers is denoted $\mathbb{C}$.

Geometrically, we think of $\mathbb{C}$ as the usual xy -plane $\mathbb{R}^2$: that is, we identify $x+iy \in \mathbb{C}$ with the point $(x, y) \in \mathbb{R}^2$. In particular, since $i = 0+1i$, we identify i with $(0, 1)$.

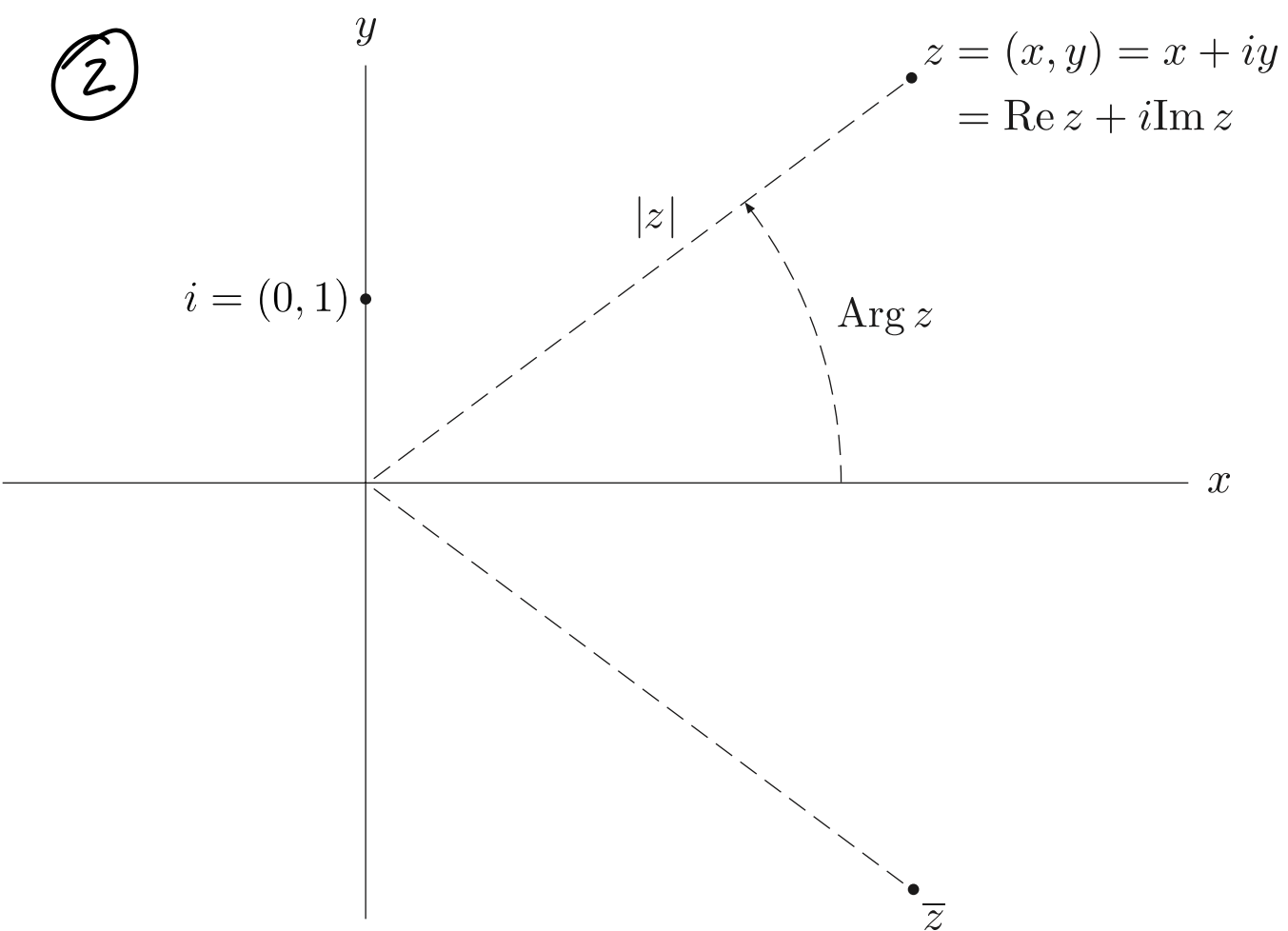
(b) Let $z = x+iy \in \mathbb{C}$. Then:

- We call x the real part of z , denoted $\operatorname{Re} z$, and call y the imaginary part of z , denoted $\operatorname{Im} z$.

- The modulus $|z|$ is the distance from z to $(0, 0)$. So $|z| = \sqrt{x^2 + y^2}$.

Also, the angle that z makes with the positive x -axis is called the argument of z , denoted $\operatorname{Arg} z$. We assume $\operatorname{Arg} z \in (-\pi, \pi]$. Finally, we denote by $\bar{z}$ the reflection of z about the x -axis. So $\bar{z} = x-iy$. We call $\bar{z}$ the complex conjugate of z .

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(c) We define addition and multiplication in $\mathbb{C}$ by way of the operations on $\mathbb{R}$, keeping in mind that $i^2 = -1$:

$$\begin{aligned}(x+iy) + (u+iv) &= (x+u) + i(y+v); \\ (x+iy)(u+iv) &= xu + x(iv) + iyv + (iy)(iv) \\ &= (xu - yv) + i(xv + yu).\end{aligned}$$

Exercise 1: Using the above definitions, show that

(a) $z + \bar{z} = 2 \operatorname{Re} z$; $z - \bar{z} = 2i \operatorname{Im} z$;

(b) $z \bar{z} = |z|^2$;

(c) If $z \neq 0$ then the unique complex number z^{-1} (also written $1/z$) such that $z z^{-1} = z^{-1} z = 1$ is given by $z^{-1} = (x-iy)/(x^2+y^2)$.

Part II: the function e^{ix} .

Definition: if $x \in \mathbb{R}$, we define the complex exponential function e^{ix} by

$$e^{ix} = \cos x + i \sin x.$$

Theorem 1: properties of e^{ix} .

We have:

(a) $|e^{ix}| = 1$.

(b) $e^{ix_1} e^{ix_2} = e^{i(x_1+x_2)}$

(c) $1/e^{ix} = e^{-ix}$ (where e^{-ix} means $e^{i(-x)}$).

(d) $(e^{ix})^n = e^{inx}$ for $n \in \mathbb{N}$.

(e) $e^{in\pi} = (-1)^n$ for $n \in \mathbb{N}$.

Proof:

(a) $|e^{ix}| = |\cos x + i \sin x| = \sqrt{\cos^2 x + \sin^2 x} = 1$.

(b) $e^{ix_1} e^{ix_2} = (\cos x_1 + i \sin x_1)(\cos x_2 + i \sin x_2)$

$$\begin{aligned}
&= \cos x_1 \cos x_2 + i \sin x_1 \cos x_2 + i \cos x_1 \sin x_2 \\
&\quad + i^2 \sin x_1 \sin x_2 \\
&= (\cos x_1 \cos x_2 - \sin x_1 \sin x_2) \\
&\quad + i(\sin x_1 \cos x_2 + \cos x_1 \sin x_2) \\
&\stackrel{\text{trig identities}}{=} \cos(x_1 + x_2) + i \sin(x_1 + x_2) \\
&= e^{i(x_1 + x_2)}.
\end{aligned}$$

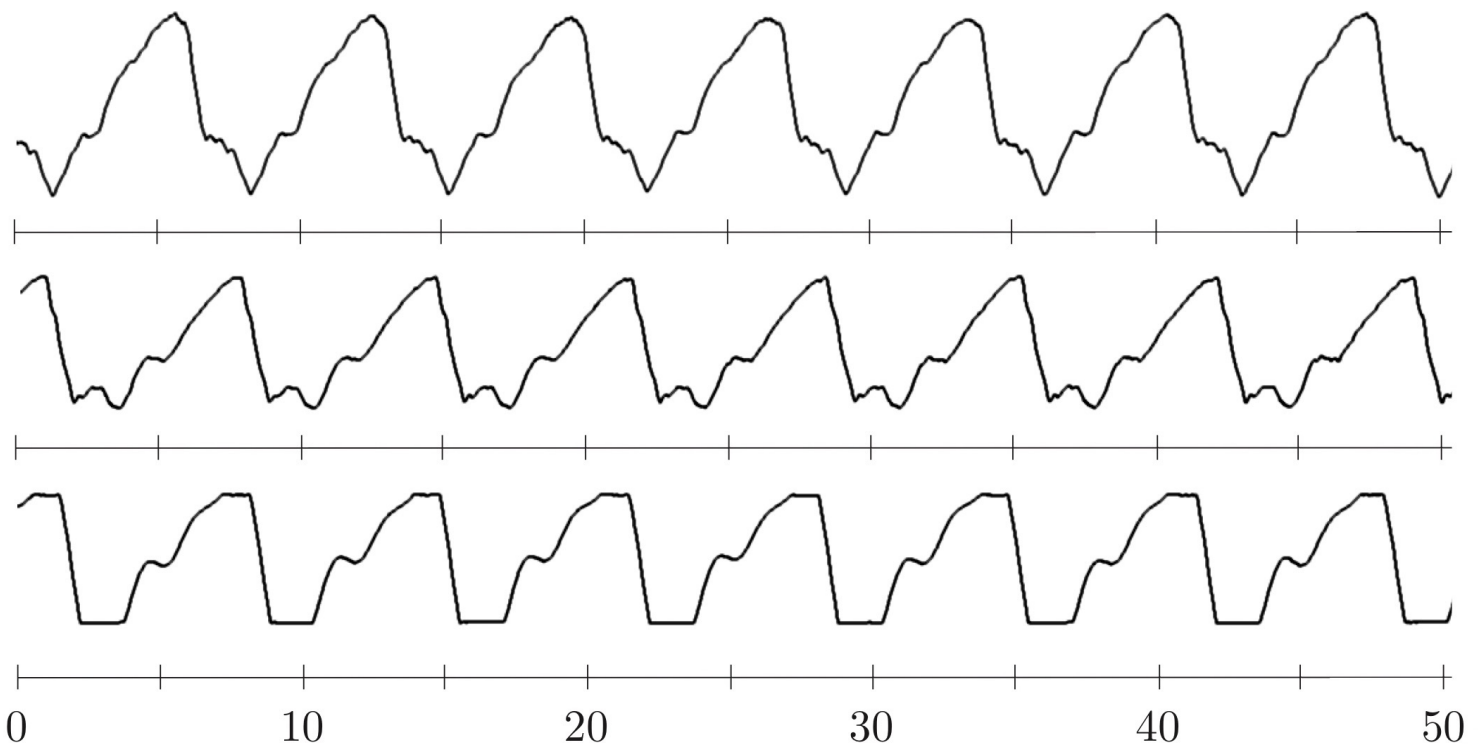
Parts (c)(d)(e): this is Exercise 2. $\square$

Part III: periodic functions.

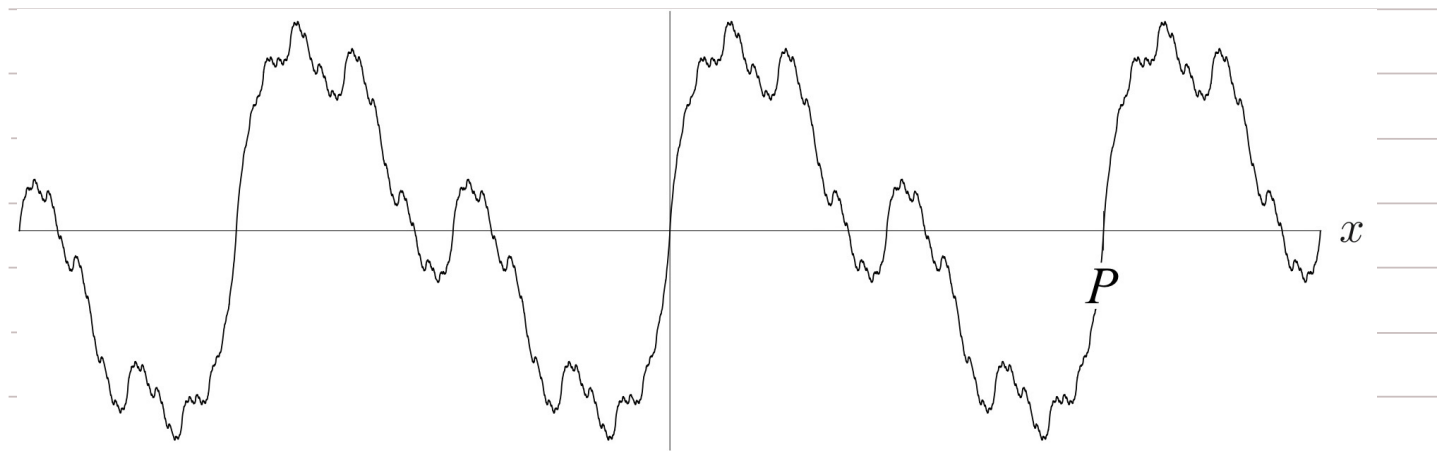
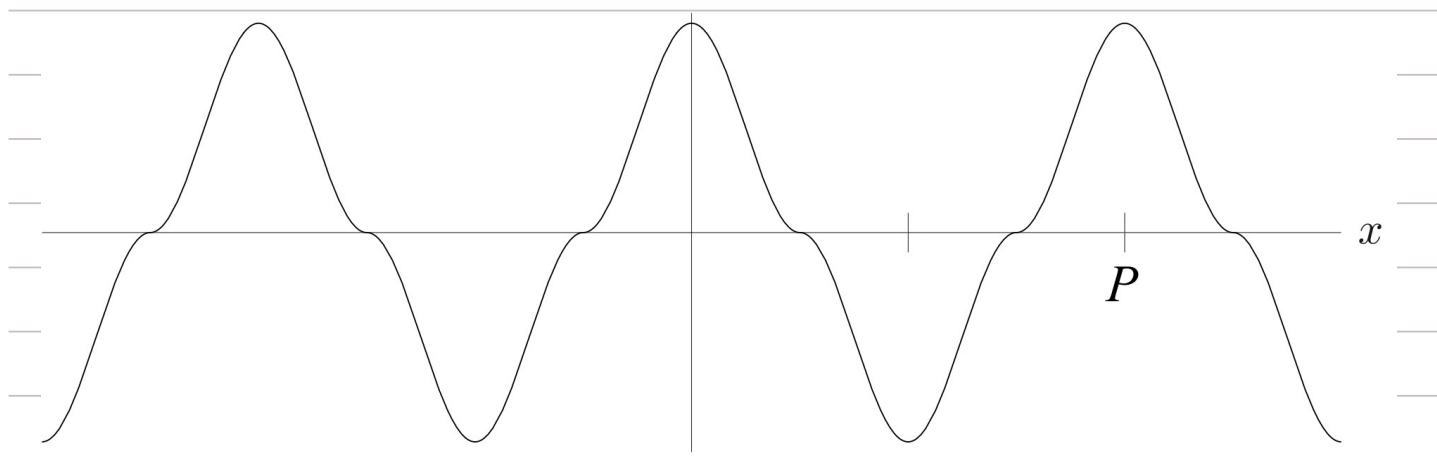
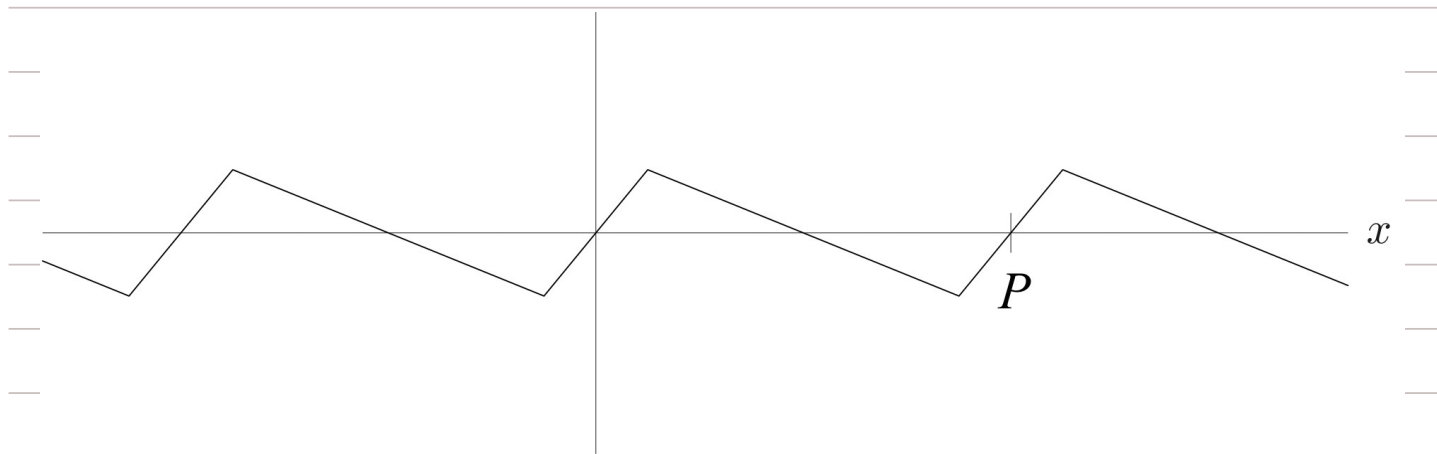
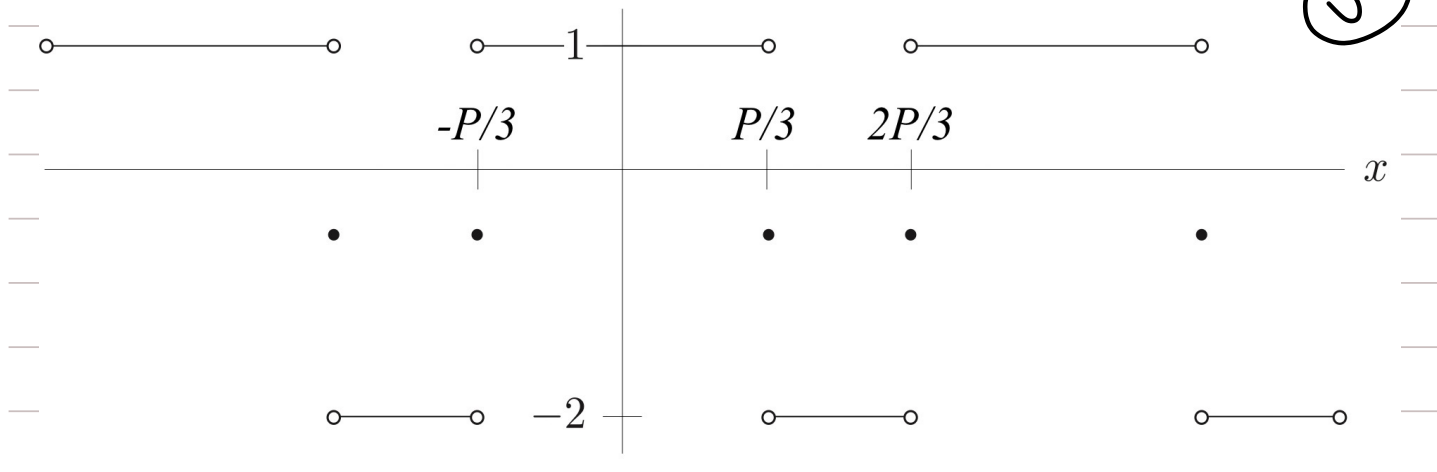
Definition 2. Let $P > 0$.

A function $f: \mathbb{R} \rightarrow \mathbb{R}$ is said to be P-periodic if $f(x+P) = f(x) \quad \forall x \in \mathbb{R}$.

A P-periodic function is one whose graph repeats itself every P units.



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For simplicity, let's focus on 2π -periodic functions.

For example, let $n \in \mathbb{Z}$. Then the function p_n defined by $p_n(x) = \cos(nx)$ is 2π -periodic, since

$$\begin{aligned} p_n(x+2\pi) &= \cos(n(x+2\pi)) \\ &= \cos(nx+2\pi n) = \cos(nx) = p_n(x). \end{aligned}$$

Similarly, $q_n(x) = \sin(nx)$ and $e_n(x) = e^{inx} = \cos(nx) + i\sin(nx)$ define 2π -periodic functions.

Now let f be 2π -periodic. If Fourier was right, and f has a trigonometric series, then it stands to reason that the trigonometric functions in that series are 2π -periodic too. So, maybe, f has an expression of the form

$$\begin{aligned} f(x) &= \sum_{n \in \mathbb{Z}} c_n(f) e_n(x) = \sum_{n \in \mathbb{Z}} c_n(f) e^{inx} \\ &= \sum_{n \in \mathbb{Z}} c_n(f) [\cos(nx) + i\sin(nx)], \quad (*) \end{aligned}$$

for appropriate numbers $c_n(f) \in \mathbb{R}$ (or $\mathbb{C}$).

More on (*) next time.

Fourier series, concluded.

The story so far:

If f is a "reasonable" 2π -periodic function, then it's "reasonable" to expect that f has a trig series in terms of the 2π -periodic trig functions

$$e_n(x) = e^{inx} = \cos(nx) + i \sin(nx),$$

where $i = -1$ and $n \in \mathbb{Z}$.

That is, we might expect that

$$f(x) = \sum_{n \in \mathbb{Z}} c_n(f) e^{inx} \quad (*)$$

for appropriate "Fourier coefficients" $c_n(f)$.

Two BIG questions:

(1) Is this assumption correct, and if so, how "reasonable" does f have to be for $(*)$ to hold?

(2) If $(*)$ DOES hold, what IS $c_n(f)$??

It's easiest to answer (2) first, like this.

Suppose $(*)$ is true. Let's multiply both sides of $(*)$ by e^{-ikx} , where $k \in \mathbb{Z}$, then integrate both sides over $[-\pi, \pi]$, then divide both sides by 2π . We get:

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-ikx} dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\sum_{n \in \mathbb{Z}} c_n(f) e^{inx} \right) e^{-ikx} dx$$

pull stuff that doesn't depend on x outside the integral

$$= \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} c_n(f) \int_{-\pi}^{\pi} e^{i(n-k)x} dx. \quad (*')$$

(2)

Now:

(a) If $n=k$, then $e^{i(n-k)x} = e^{i0x} = e^{i0} = \cos 0 + i \sin 0 = 1$, so the integral on the right side of $(*)'$ equals

$$\int_{-\pi}^{\pi} 1 dx = 2\pi.$$

(b) If $n \neq k$, then the integral on the right side of $(*)'$ equals

$$\int_{-\pi}^{\pi} e^{-i(n-k)x} dx = \frac{e^{-i(n-k)x}}{-i(n-k)} \Big|_{-\pi}^{\pi}$$

antidifferentiate like

i is any other constant

$$= \frac{e^{-i(n-k)\pi} - e^{-i(n-k)(-\pi)}}{-i(n-k)} = \frac{(-1)^{k-n} - (-1)^{n-k}}{-i(n-k)} = 0 !!$$

Exercise 2d
last time

So $(*)'$ gives

$$\frac{1}{2\pi i} \int_{-\pi}^{\pi} f(x) e^{-ikx} dx = \frac{1}{2\pi} \cdot c_k(f) \cdot 2\pi = c_k(f).$$

That is, replacing k by n ,

$$c_n(f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx.$$

CONCLUSION:

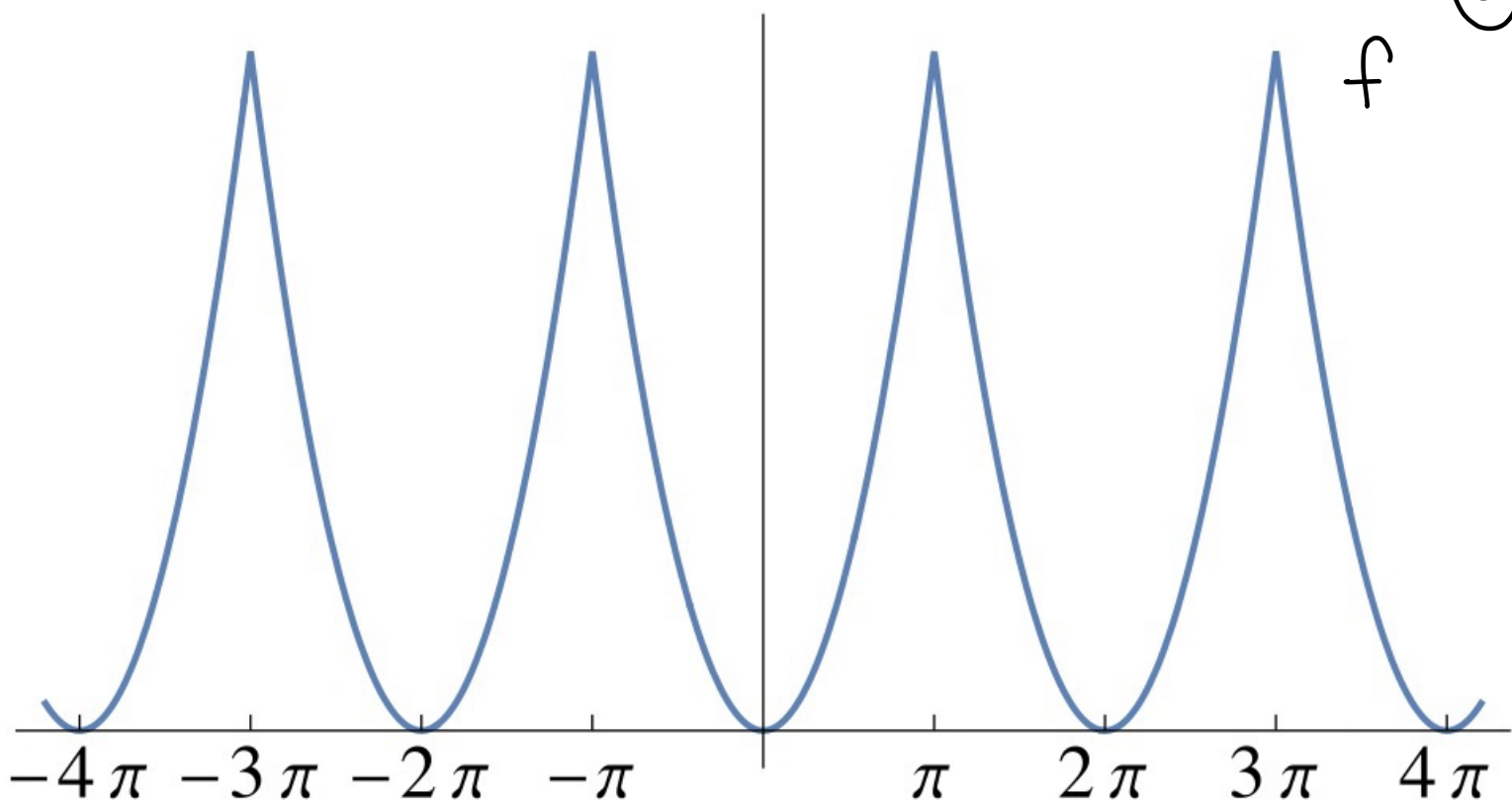
If f is 2π -periodic and has a Fourier series $(*)$,
then

$$c_n(f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-ikx} dx. \quad (**)$$

Exercise 1. Let f be the 2π -periodic function defined on $[-\pi, \pi]$ by $f(x) = x^2$.

③

f



Show that

$$c_n(f) = \begin{cases} \pi^2/3 & \text{if } n=0, \\ 4\pi(-1)^n/n^2 & \text{if } n \neq 0. \end{cases}$$

Partial solution.

We have, by (**),

$$c_n(f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} x^2 e^{-inx} dx.$$

Now we use (without proof) the formula

$$\int x^2 e^{ax} dx = \begin{cases} x^3/3 + C & \text{if } a=0, \\ \frac{e^{ax}(a^2 x^2 - 2ax - 2)}{a} + C & \text{if } a \neq 0 \end{cases}$$

(where a is any constant $\neq 0$) to finish.

FINALLY, we address, without proof, question (1):
WHEN (for which f) does (**) hold?

We have the following, due to Dirichlet et. al.:

Theorem 1.

If f is 2π -periodic and continuous on $\mathbb{R}$
and f' is piecewise continuous on $\mathbb{R}$, then

$$f(x) = \sum_{n \in \mathbb{Z}} c_n(f) e^{inx} \quad \forall x \in \mathbb{R}.$$

Exercise 2.

Use Exercise 1 and the above theorem to show
that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Hint: use Theorem 1 with $x=\pi$.