More induction.

Example 1: Fibonacci numbers.

The Fibonacci numbers $F_1, F_2, F_3, \ldots$ are defined by

$F_1 = 1$, $F_2 = 1$, and (initial values)

$F_{n+2} = F_{n+1} + F_n$ for $n \geq 1$. (recurrence relation)

[So the first few F_n's are

$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233 \ldots$.]

Show that, $\forall n \in \mathbb{N}$,

$$F_1 + F_2 + F_3 + \ldots + F_n = F_{n+2} - 1.$$

Solution.

Let $A(n)$ be the statement in question.

Is $A(1)$ true?

$$F_1 \stackrel{?}{=} F_3 - 1$$

$$1 = 2 - 1 = 1 \checkmark$$

So $A(1)$ is true.

Now assume

$$A(k): F_1 + F_2 + \ldots + F_k = F_{k+2} - 1.$$

Then

$F_1 + F_2 + \ldots + F_{k+1}$

$= (F_1 + F_2 + \ldots + F_k) + F_{k+1}$ ("break off" left side of $A(k)$)

$= F_{k+2} - 1 + F_{k+1}$ (by the induction hypothesis)

$= F_{k+2} + F_{k+1} - 1$ (rearrange)

$= F_{k+3} - 1$, (by the recurrence relation)

so $A(k+1)$ follows.

So by induction, $A(n)$ is true $\forall n \in \mathbb{N}$.

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Example 2: divisibility.

Show that, $\forall n \in \mathbb{Z}_{\geq 0} = \{0,1,2,3,\ldots\}$, 7 divides $12^n - 5^n$.

Solution. Let $A(n)$ be the statement in question, which says

$$12^n - 5^n = \text{a multiple of } 7.$$

Is $A(0)$ true?

$$12^0 - 5^0 \overset{?}{=} \text{a multiple of } 7$$

$$1 - 1 = 0 = 7 \cdot 0,$$

so $A(0)$ is true.

Now assume

$$A(k): 12^k - 5^k = 7m \text{ for some } m \in \mathbb{Z}.$$

Then

$$\begin{aligned}
&12^{k+1} - 5^{k+1} \\
&= 12 \cdot 12^k - 5 \cdot 5^k && \text{(rewrite so exponents are k's)} \\
&= 12 \cdot 12^k - (12-7) \cdot 5^k && \text{(trick to get a 7 somewhere)} \\
&= 12 \cdot 12^k - 12 \cdot 5^k + 7 \cdot 5^k && \text{(distributive law)} \\
&= 12 \cdot (12^k - 5^k) + 7 \cdot 5^k && \text{(regroup)} \\
&= 12 \cdot 7m + 7 \cdot 5^k && \text{(induction hypothesis)} \\
&= 7(12^m + 5^k), && \text{(regroup)}
\end{aligned}$$

so $12^{k+1} - 5^{k+1}$ is also a multiple of 7.

So $A(k) \Rightarrow A(k+1)$.

So by induction, $A(n)$ holds $\forall\, n \in \mathbb{N}$. □

3) Variants on induction:

(a) "Induction by strides:" instead of a single base case, there are M of them (for some fixed $M \in \mathbb{N}$). Then the induction step is to show $A(k) \Rightarrow A(k+M)$.

Example: to show that every integer $n \geq 8$ can be written as $3a+5b$ for a, b integers ≥ 0:

(i) first show it's true for $n=8$, $n=9$, and $n=10$.

(ii) then show that, if it's true for $k \geq 8$, then it's true for $k+3$.

(b) Strong induction: instead of assuming $A(k)$, you assume $A(1), A(2), A(3), \ldots A(k)$.

Example: to show that every integer $n \geq 2$ is a product of (one or more) primes:

(i) First show it's true for $n=2$.

(i) Then show that, if it's true for the integers $2, 3, \ldots, k$, then it's true for $k+1$.