

The Principle of Mathematical Induction.

(A proof template for statements of the form

$$\forall n \in \mathbb{N}: A(n).$$

Idea: let $A(n)$ be a statement about a natural number n .

Example:

$$1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}.$$

Suppose we can show that:

(1) $A(1)$ is true, and

(2) whenever $A(k)$ is true for some $k \in \mathbb{N}$, $A(k+1)$ follows. In other words,
 $\forall k \in \mathbb{N}, A(k) \Rightarrow A(k+1).$

Then by (1), $A(1)$ holds, so by (2), $A(2)$ holds, so by (2), $A(3)$ holds, so by (2), $A(4)$ holds... so "ultimately," $A(n)$ holds for any $n \in \mathbb{N}$.

So we have a "mathematical induction" proof template:

Theorem. $\forall n \in \mathbb{N}, A(n).$

Proof

Step 1: Is $A(1)$ true? [Prove $A(1).$]

So $A(1)$ is true.

Step 2: Assume $A(k)$. [Now do what's

necessary to conclude:] So $A(k+1)$ follows.

So by (the principle of mathematical induction, $A(n)$ is true $\forall n \in \mathbb{N}_0$.

□

Remarks

Step 1 is the "base step".

Assuming $A(k)$ is the "induction hypothesis".

Deducing $A(k+1)$ is the "inductive step".

Step 2

Example 1:

Prove that, $\forall n \in \mathbb{N}$,

$$1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}.$$

Proof.

Let $A(n)$ be the statement

$$1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}.$$

Step 1: is $A(1)$ true?

$$1 \stackrel{?}{=} \frac{1(1+1)}{2}$$

$$1 = 1 \quad \checkmark$$

So $A(1)$ is true.

Step 2: Assume

$$A(k): 1 + 2 + 3 + \dots + k = \frac{k(k+1)}{2}.$$

To deduce $A(k+1)$, we note that

write out the left
side of $A(k+1)$

express it in terms
of the left side of $A(k)$

$$1 + 2 + 3 + \dots + k + 1 = (1 + 2 + 3 + \dots + k) + k + 1$$

(3)

↙ invoke (use) the induction hypothesis

$$= \frac{k(k+1)}{2} + k+1$$

↘ do stuff to get to the right side of $A(k+1)$.

$$= \frac{k(k+1)}{2} + \frac{2(k+1)}{2} = \frac{k(k+1) + 2(k+1)}{2} = \frac{(k+1)(k+2)}{2}.$$

So $A(k+1)$ follows.

So by induction,

$$1+2+3+\dots+n = \frac{n(n+1)}{2} \quad \forall n \in \mathbb{N}. \quad \square$$

Tips for the inductive step (in many cases):

- write out the left side of $A(k+1)$ *
- Express this in terms of the left side of $A(k)$
- Invoke the induction hypothesis.
- Turn the result into the right side of $A(k+1)$.

* in scratch somewhere, maybe write out the right side as well, so you know what to aim for