Quantifier proofs, continued.

Example

Let $f(x) = 2x-5$.

Prove that

$\forall \varepsilon > 0$*, $\exists \delta > 0 \ni$

$$0 < |x-4| < \delta \Rightarrow |f(x) - 3| < \varepsilon.$$

*that is, $\varepsilon \in \mathbb{R}^+ = \{\text{positive reals}\}$.

Solution.

(not an "official" part of the proof)

Let $\varepsilon > 0$. [Scratchwork: we want $|f(x)-3| = |2x-5-3| = |2x-8| < \varepsilon$. How small should $|x-4|$ be?

Well, $|2x-8| = 2|x-4|$, so we want $2|x-4| < \varepsilon$, meaning $|x-4| < \varepsilon/2$.

So $\delta = \varepsilon/2$ works. Then write this:] Let $\delta = \varepsilon/2$. Then

$$0 < |x-4| < \delta \Rightarrow |f(x)-3| = |2x-5-3| = |2x-8|$$
$$= 2|x-4| < 2\delta = 2 \cdot \varepsilon/2 = \varepsilon.$$

So $\forall \varepsilon > 0$, $\exists \delta > 0 \ni |x-4| < \delta \Rightarrow |2x-5| < \varepsilon$.

$\square$