

Homework Assignment 1: Solutions to Selected Exercises

Please note that starred problems in the Exercises have hints and/or solutions at the back of the book.

Assignment from the text:

Section 1.1: Exercises 3, 4, 9, 10, 11, 12.

Section 1.2: Exercises 1, 2, 4, 5, 7, 10, 11, 13, 14, 17, 18.

Section 1.3: Exercises 1–5, 6acefgk, 7adgh, 8.

Section 1.4: Exercises 1, 2, 3, 6, 9, 11, 17, 18, 24abc, 31.

Section 1.1

4. (a) The function $f(x) = x^2 - 9$ is not continuous at $x = 3$. (b) The relation R is neither reflexive nor symmetric. (c) Four and nine are not relatively prime. (d) x is not in A and x is in B . (e) There is some $x < 7$ such that $f(x)$ is in C . (f) There is a convergent sequence (a_n) that is either not monotone or not bounded. (g) There exist a continuous function f and an open set A such that $f^{-1}(A)$ is not open.

10. (a) False. (b) False. (c) True. (d) False. (e) True. (f) True. (g) False. (h) True. (i) False. (j) True.

12. (a) $x \perp N \wedge x \not\perp M$. (b) $x \not\perp M \wedge x \not\perp N$. (c) $x \perp N \Rightarrow x \perp M$. (d) $x \perp M \Rightarrow x \not\perp N$. (e) $\sim (x \perp M \wedge x \perp N)$.

Section 1.2:

1. (a) True. (b) False. (c) True.

2. (a) False. (b) True. (c) True.

4. (a) Not everyone likes Robert. (b) No students work part-time. (c) There exists a square matrix that is triangular. (d) $\forall x$ in B , $f(x) \leq k$. (e) There exists an $x < 5$ such that $f(x) \geq 3$ and $f(x) \leq 7$. (f) There exists an x in A such that, for all $y \in B$, $f(x) \geq f(y)$.

7. (c) (ii). (d) (i).

14. (a) A function f is *periodic* if $\exists k > 0 \ni \forall x, f(x+k) = f(x)$. (b) A function f is *not periodic* (or *aperiodic*) if $\forall k > 0, \exists x \ni f(x+k) \neq f(x)$.

18. (a) A function $f: A \rightarrow B$ is *surjective* if $\forall y \in B, \exists x \in A \ni f(x) = y$. (b) A function $f: A \rightarrow B$ is *not surjective* if $\exists y \in B \ni \forall x \in A, f(x) \neq y$.

Section 1.3:

1. (a) True. (b) False. (c) False. (d) False. (e) True.

2. (a) True. (b) False. (c) True. (d) True. (e) False.

4. (a) If some violets are blue, then all roses are red. (b) If A is not invertible, then there exists a nontrivial solution to $Ax = 0$. (c) If $f(C)$ is connected, then f is continuous and C is connected.

5. (a) If not all roses are red, then no violets are blue. (b) If there is no nontrivial solution to $Ax = 0$, then A is invertible. (c) If f is discontinuous or C is not connected, then $f(C)$ is not connected.

6. (a) $x = -4$. (c) $x = 0.3$. (e) $n = 41$. (f) The number 2 is a counterexample. (g) The integer 103 is a counterexample. (k) $x = 0$.

7. (a) Suppose that p is odd and q is odd. Then we can write $p = 2k + 1$ and $q = 2\ell + 1$, where $k, \ell \in \mathbb{Z}$. Then

$$p + q = 2k + 1 + 2\ell + 1 = 2(k + \ell + 1).$$

Since $k + \ell + 1 \in \mathbb{Z}$, $p + q$ is even.

So p, q odd $\Rightarrow p + q$ is even. $\square$

(d) Suppose that p is odd and q is even. Then we can write $p = 2k + 1$ and $q = 2\ell$, where $k, \ell \in \mathbb{Z}$. Then

$$p + q = 2k + 1 + 2\ell = 2(k + \ell) + 1.$$

Since $k + \ell \in \mathbb{Z}$, $p + q$ is odd.

So p odd, q even $\Rightarrow p + q$ is odd. $\square$

(g) Suppose it's not the case that p is odd and q is odd. Then either p is even or q is even. In the first case, write $p = 2k$ where $k \in \mathbb{Z}$. Then

$$pq = (2k) \cdot q = 2(kq).$$

Since $kq \in \mathbb{Z}$, pq is even. In the second case, write $q = 2\ell$ where $\ell \in \mathbb{Z}$. Then

$$pq = p \cdot (2\ell) = 2(p\ell).$$

Since $p\ell \in \mathbb{Z}$, pq is even.

So pq odd $\Rightarrow p, q$ are odd. $\square$

8. Let $f(x) = 4x + 7$, and assume that $f(x_1) = f(x_2)$. Then $4x_1 + 7 = 4x_2 + 7$. Subtracting 7 from both sides, and then dividing both sides of the result by 4, we find that $x_1 = x_2$.

So, if $f(x) = 4x + 7$, then $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$. $\square$

Section 1.4:

1. (a) True. (b) True. (c) True.

2. (a) True. (b) False. (c) True.

3. Let $\varepsilon > 0$. [Scratchwork: we want $11 - \varepsilon < 3x + 5 < 11 + \varepsilon$, or equivalently $|3x + 5 - 11| < \varepsilon$. But

$$|3x + 5 - 11| = |3x - 6| = 3|x - 2|.$$

So it's enough to choose $3|x - 2| < \varepsilon$, meaning $|x - 2| < \varepsilon/3$. So this is what we write:] Let $\delta = \varepsilon/3$. Then

$$|x - 2| < \delta \Rightarrow |3x + 5 - 11| = |3x - 6| = 3|x - 2| < 3 \cdot \varepsilon/3 = \varepsilon,$$

so $11 - \varepsilon < 3x + 5 < 11 + \varepsilon$.

So for every $\varepsilon > 0$, $\exists \delta > 0$ $\ni$

$$2 - \varepsilon < 7 - 5x < 2 + \varepsilon \Rightarrow 11 - \varepsilon < 3x + 5 < 11 + \varepsilon. \quad \square$$

6. Let $x = 0$. Then for any real number y , $xy = 0 \cdot y = 0 = x$. So there exists a real number x such that for every real number y , $xy = x$. $\square$

11. Let $x > 5$. Let $y = \frac{3x}{5-x}$. Note that $y < 0$ since $x > 5$. Also note that we can solve for x to find that $x = 5y/(y + 3)$. So $\forall x > 5, \exists y < 0 \ni x = 5y/(y + 3)$. $\square$

17. (a) This is a proof of the converse of the given statement, not the statement itself. (b) This is a proof of the contrapositive of the given statement, and is therefore OK.

18. (a) This is OK. (b) This is OK.

24. (a) $a^2 + b^2 \neq c^2$. (b) $a^2 + b^2 \neq c^2$. (c) $2k + 3, 2k + 3, 15, -3/2$. Integer.

31. $x = -1$ is a counterexample, since then $x^2 + 1 = 2$, which is not less than $(x + 1)^2 = 0$.